Non-Euclidean Erdős–Anning Theorems

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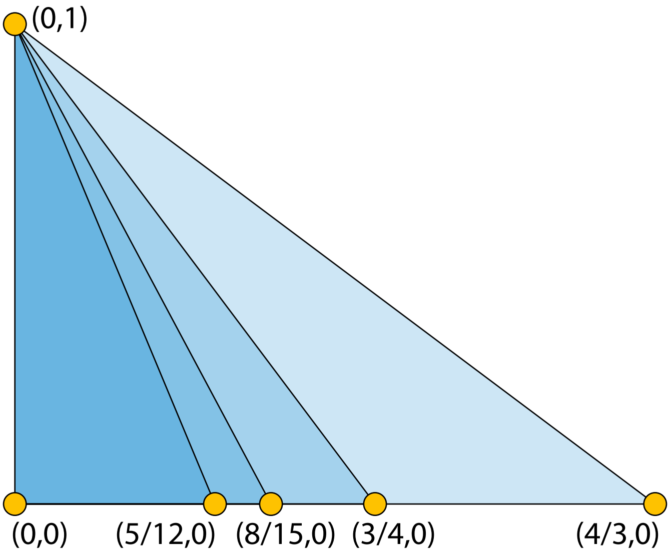
In one dimension

Easy to find infinitely many points at integer distances: $\mathbb{Z}$

...or less abstractly, milestones along a road

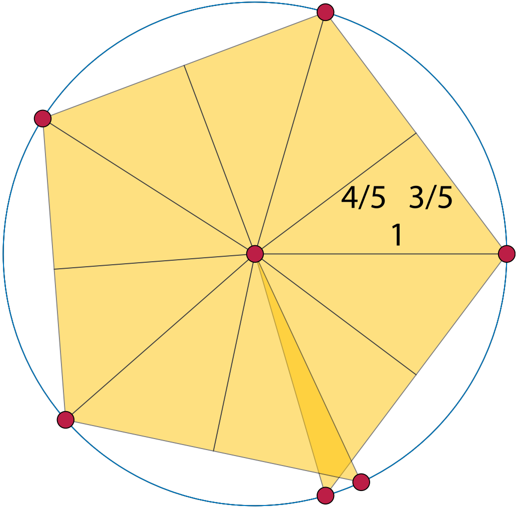
All the Pythagorean triangles scaled to share a side



$\Rightarrow$ infinitely many points, not all on one line, with rational distances

Yet another result of Euler

There exist infinitely many points at rational distances on a unit circle

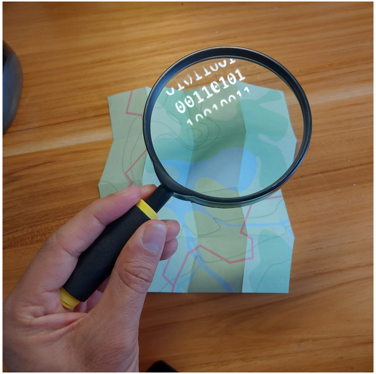


But what about integers?

Scaling can turn any finite set of points with rational distances into a set with integer distances

But more points $\Rightarrow$ bigger common denominator of distances $\Rightarrow$ bigger scale factor

So we can get arbitrarily large but not infinite non-collinear integer distance sets



The Erdős–Anning theorem

Anning and Erdős [1945]:
Every integer distance set in $\mathbb{R}^2$ is either:
finite, or
collinear

(messy trigonometric proof)

Erdős [1945]:
Non-collinear set with diameter D has $O(D^2)$ points

(five-line proof)

"An analogous theorem clearly holds in higher dimensions."

points subsequently improved to $D^{O(1/\log\log D)}$ [Greenfeld et al. 2024]

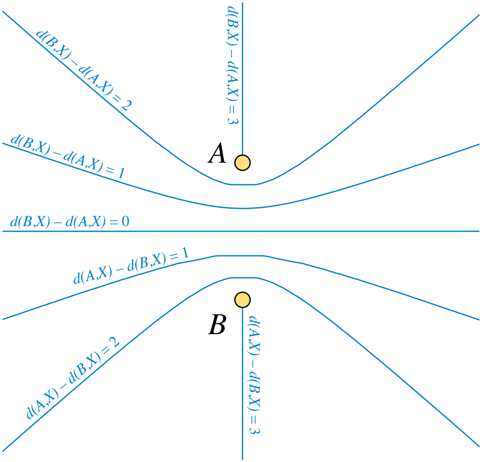
Points at integer distance from two fixed points

Given two points at integer distance D

Integer distances to any third point must differ by an integer $\delta \leq D$

Each choice of $\delta \Rightarrow$ a hyperbola

Or, opposite rays (a degenerate hyperbola)



Erdős's proof

If non-collinear $\{A, B, C\} \subset S$:

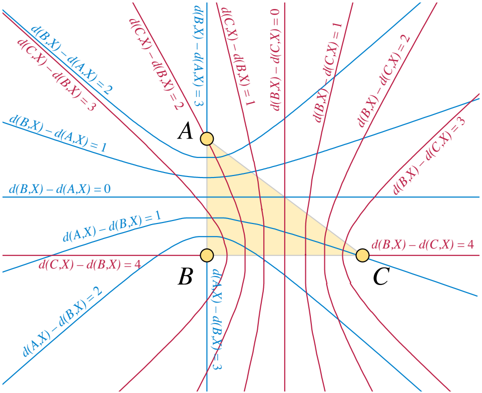
Overlay $\leq D + 1$ hyperbolas for AB and another $\leq D + 1$ for BC

All points at integer distances must belong to both systems of hyperbolas

No two hyperbolas can coincide (omitted as obvious by Erdős)

Two distinct hyperbolas cross at most four times (because they are quadratic algebraic curves, by Bezout's theorem)

$S \subset$ crossings $\Rightarrow |S| \leq 4(D + 1)^2$



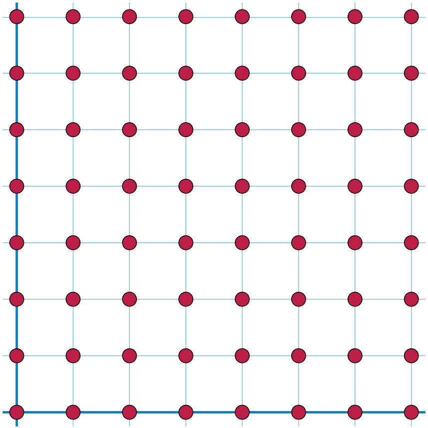
What ingredients are needed for non-Euclidean distances?

Algebraic curves (hyperbolas) and intersection theory (Bezout's theorem) are very specific to Euclidean distance

⇒ need topological proof, not algebraic

Erdős–Anning untrue for L_1 and L_∞: they have infinite non-collinear integer distance sets

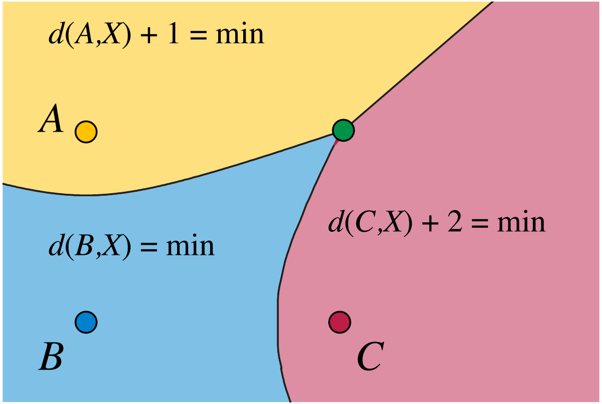
⇒ need careful analysis of distance properties



Three sites are better than two

For three fixed sites A, B, C and a variable point X at integer distances from all three, add integers to distances to A, B, C to make offset distances to X become equal

$\Rightarrow X \in$ triple intersection of (closed) cells of additively-weighted Voronoi diagram of A, B, and C; Erdős's hyperbolae become cell boundaries



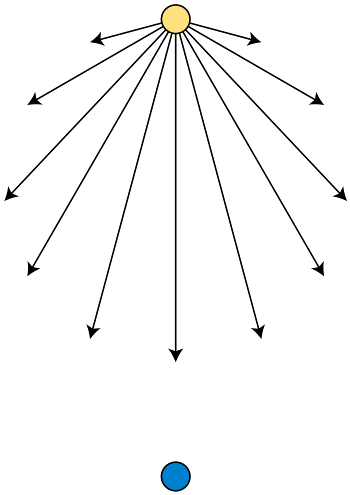
So let's try to understand cell intersections in additively-weighted Voronoi diagrams

Euclidean additively weighted Voronoi cells are star-shaped

Moving straight towards a site, length traveled = reduction in distance to site

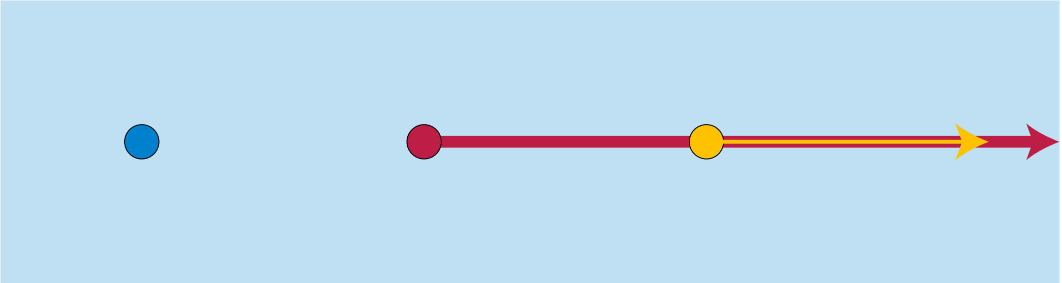
Distance to any other site cannot decrease more quickly

So along line segment to closest site, it remains closest until site reached



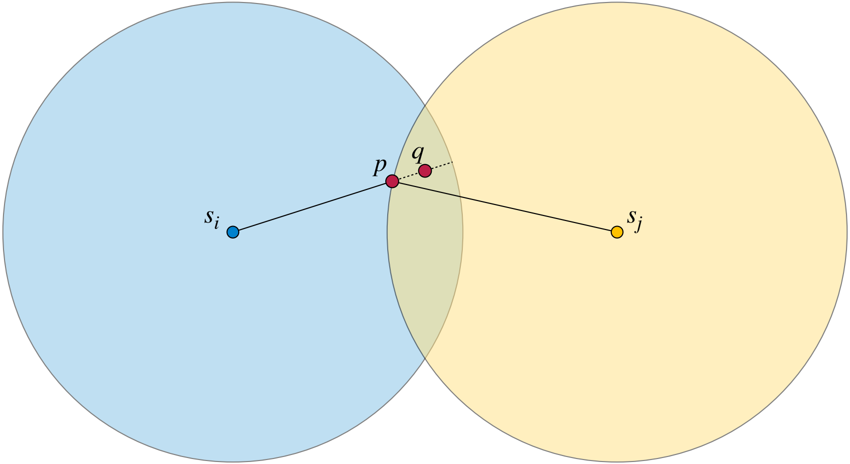
But they can be degenerate!

If weighted distance from y to itself = weighted distance to another site x
then Voronoi cell of $y \subset$ ray in opposite direction from x
but this ray is a subset of the cell for x!



Degenerate Voronoi cells can nest for collinear sites $\Rightarrow$ # triple intersections $= \infty$

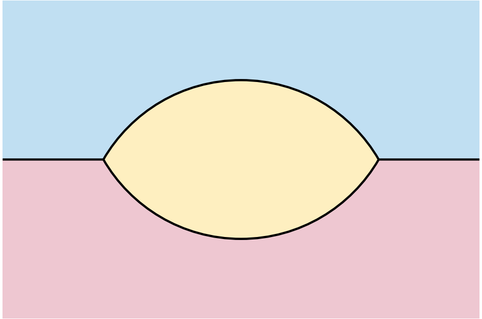
Non-degenerate cells are interior-disjoint



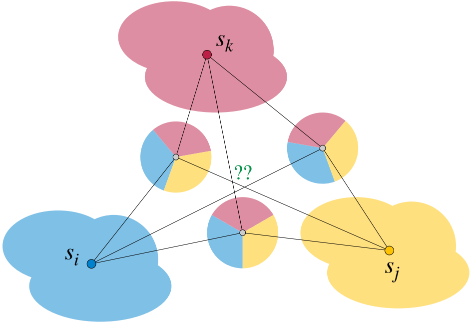
If interior point p of cell for s_i could also belong to cell for s_j,
bent path q–p–s_j would be a shortest path, impossible

Three non-degenerate cells have ≤ 2 triple intersections

Two intersections is possible

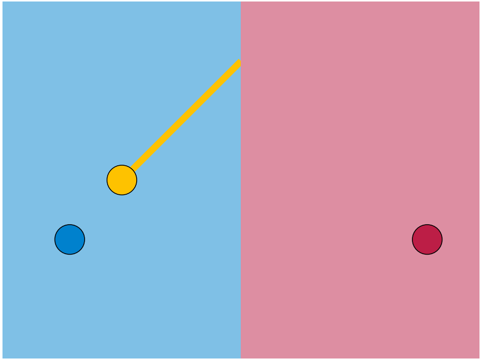


Three would give a planar drawing of $K_{3,3}$



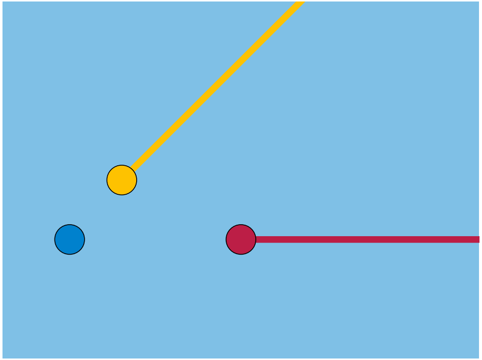
Degenerate but non-collinear cases

One degenerate cell



One triple intersection

Two degenerate cells



No triple intersections

New proof for the old Euclidean distance

For non-collinear points with integer distances:

- Choose any non-collinear triple
- Let D be their max distance
- For each weighting where one weight is zero and the other two are $\leq D$:
  - Construct additively weighted Voronoi diagram
  - It has ≤ 2 triple points
- Every given point comes from diagram with weight $=$ distance to farthest site $-$ distance to site

$\Rightarrow$ $O(D^2)$ points, a finite number

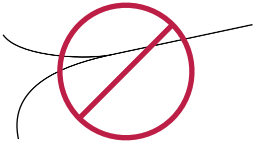
What properties do other distances need?

Erdős: equidistant curves are algebraic

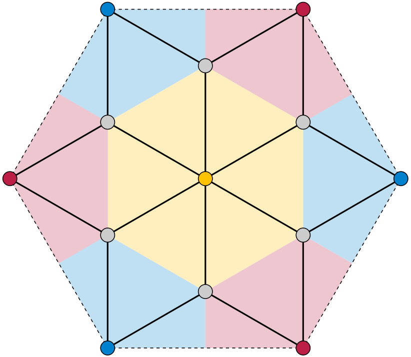
New proof:

- Geodesic: distance = length of shortest curve

  Collinearity = belong to a shortest curve
- Shortest curves should not merge or split



  (but multiple curves between the same two points are not problematic)
- Bounded Euler genus
  $\Rightarrow$ no non-crossing drawing of $K_{3,2g+3}$

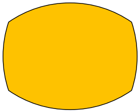


$K_{3,6}$ from three Voronoi sites and six triple intersection points on a flat hexagonal torus

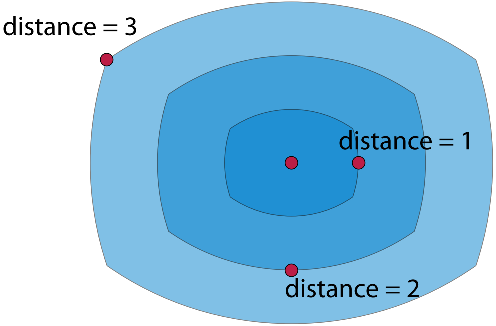
Convex distance functions

Defined from any centrally symmetric convex body

Distance(p, q) = scale factor for body centered at p to touch q



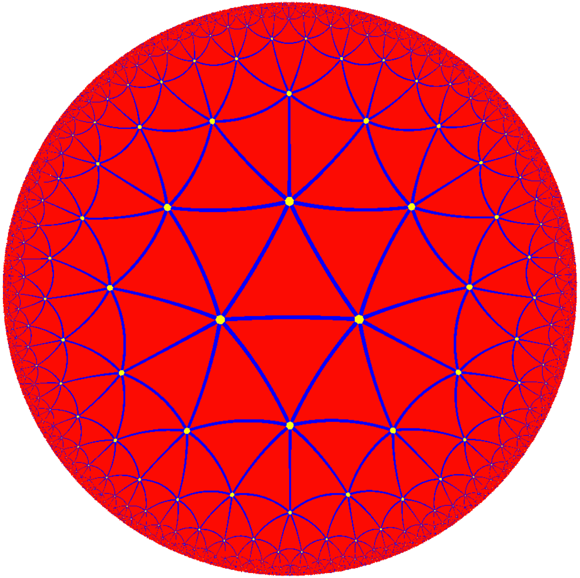
defining shape



When body is strictly convex (no line segment in boundary),

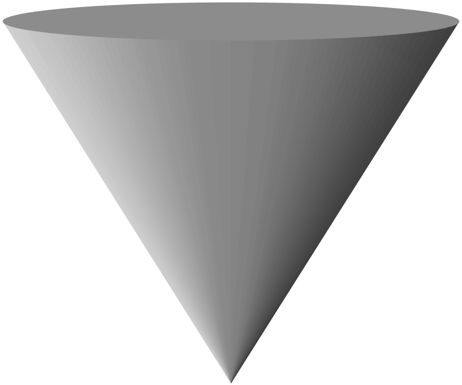
shortest curves are line segments $\Rightarrow$ all necessary properties

Hyperbolic plane



Shortest curves are hyperbolic line segments $\Rightarrow$ all necessary properties

Surfaces of convex sets in 3d



Locally approximately Euclidean except at cone points of positive curvature

Shortest curves pass straight through locally Euclidean points

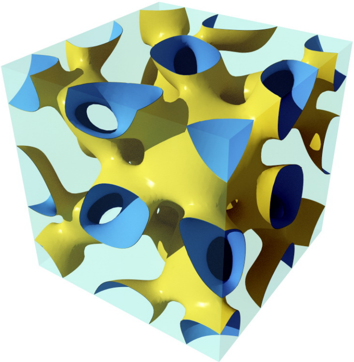
Cannot pass through cone points at all

⇒ all necessary properties

Complete Riemannian metrics

Riemannian: smooth inner product structure; locally approximately Euclidean
Complete: All Cauchy sequences converge

Required additional property: bounded genus



Includes all smooth surfaces without boundary in 3d

Not required to be convex, positively curved, or embedded in 3d

A bad locally Euclidean and incomplete Riemannian metric

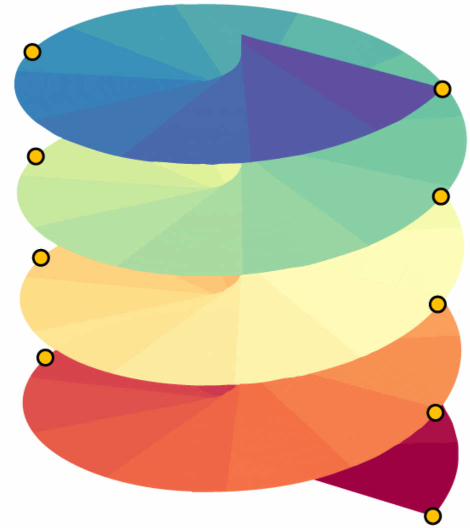
Universal cover of punctured plane

= cone with infinite angle at cone point

= polar coordinates (r, θ)
without reducing θ mod 2π

= flattened half-helicoid

Infinite unit-distance set: points with polar coordinates $(\frac{1}{2}, k\pi)$ for $k \in \mathbb{Z}$



Conclusions

Non-collinear integer distance sets must be finite for broad classes of 2d distances

An elegant proof $\neq$ a flexible proof

In paper but not in these slides:

points $= O(\text{diam})$ for strictly convex dist. func., $O(\text{diam}^{4/3})$ for convex surface

Resolution of an unsolved problem of Guy [1983] on equilateral dimension of Riemannian manifolds (max # points at unit distance)

Possible future work:

Expand Erdős's "An analogous theorem clearly holds in higher dimensions."

Unfortunately 3d Voronoi diagrams of convex distance behave badly [Icking et al. 1995]

Paper includes counterexamples for generalization to Riemannian 3-manifolds

What about three-dimensional hyperbolic space?

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References

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