Separating Thickness from Geometric Thickness

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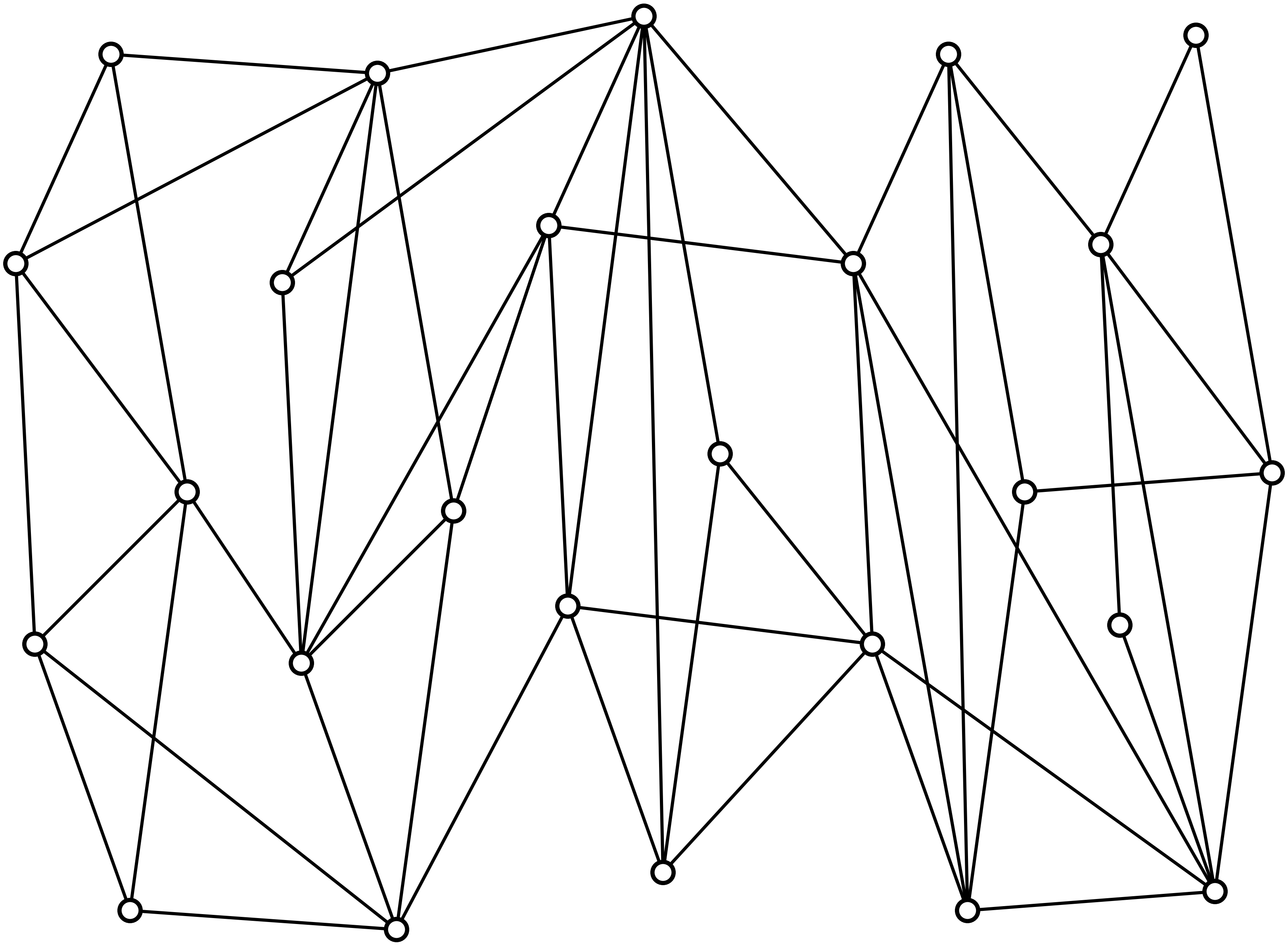
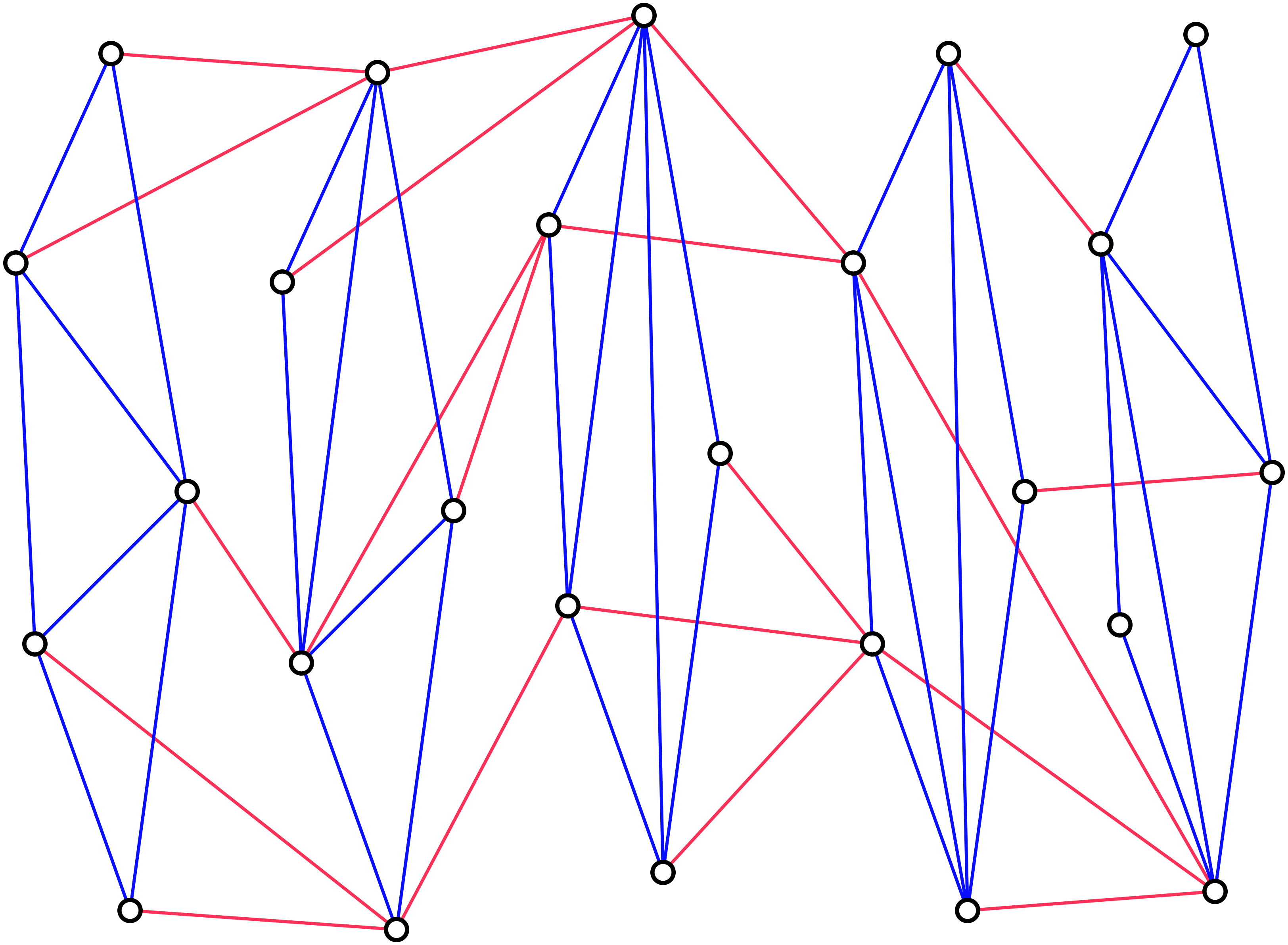
Why Thickness?

Often, graphs are nonplanar, and crossings can make edges hard to follow

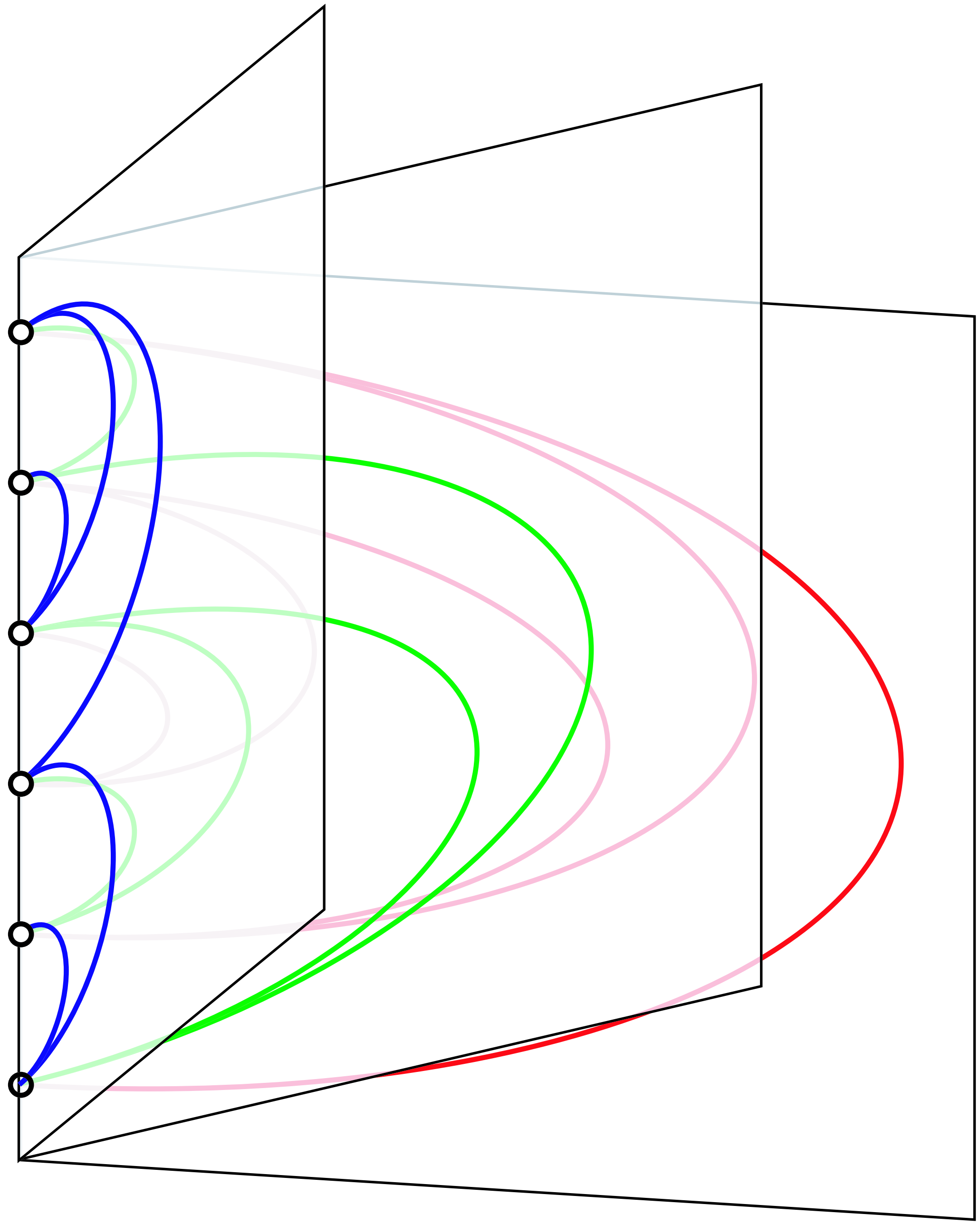


Why Thickness?

If crossing edges are given different colors, they can be easier to follow



Thickness definitions: book thickness



Place vertices vertically
along the spine of a book
(vertical line in 3d)

Place each edge into
one of the pages of the book
(open halfplane bounded by line)

Within each page, edges must
not cross each other

Can fold pages flat
to get 2d graph drawing

Edges must have bends

Thickness = min # pages needed

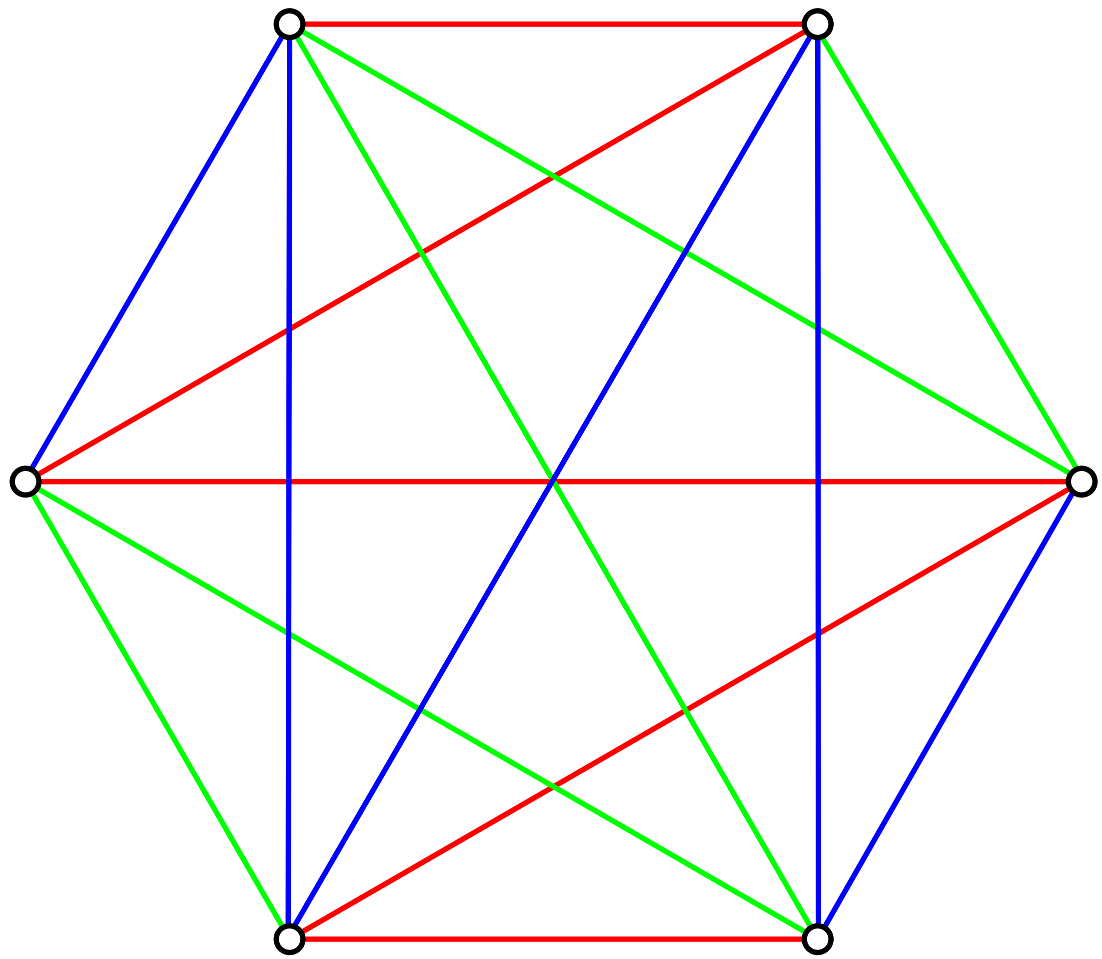
Thickness definitions: book thickness *(alternative def)*

Bend spine into a convex curve

Draw edges as straight
line segments

Use one color per page

Result: drawing with
vertices in convex position
(e.g. regular polygon),
straight edges,
edges can cross only if
they have **different colors**

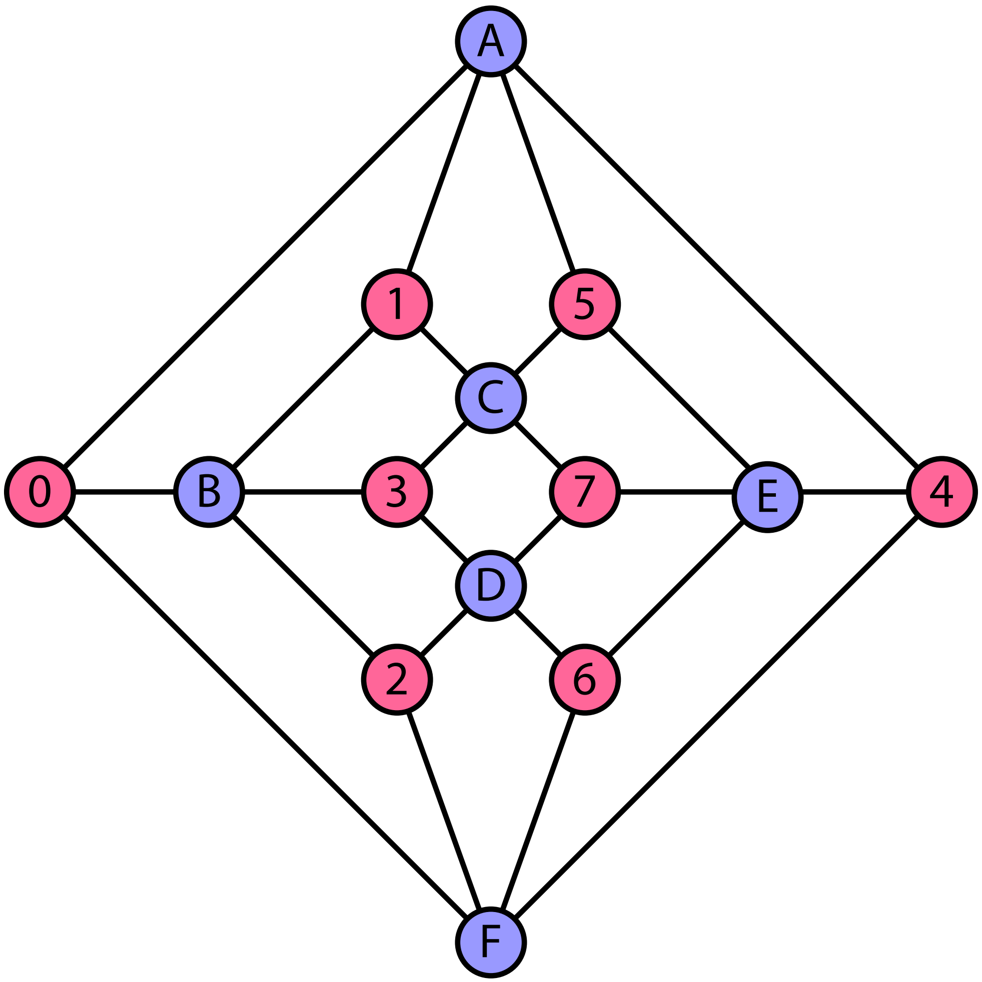


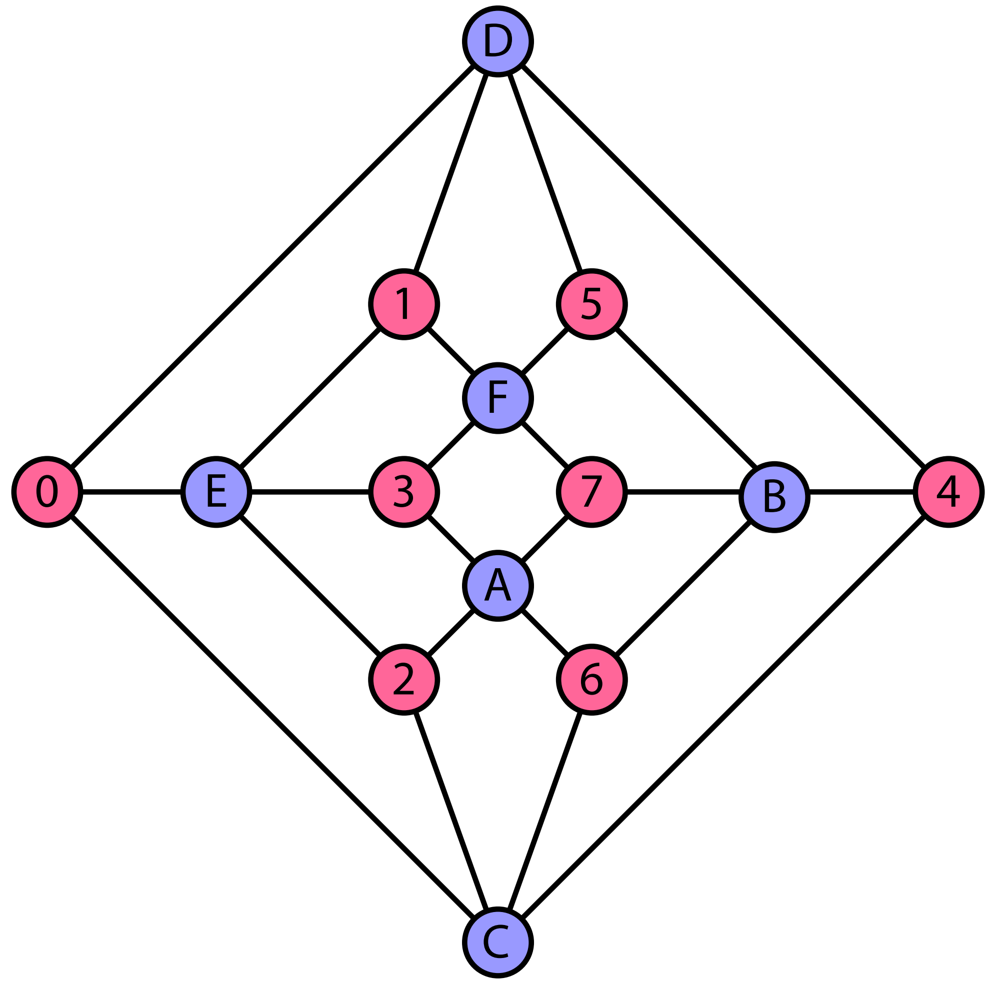
Example: K_6

Thickness definitions: graph thickness

Partition graph edges into **minimum # of planar subgraphs**

Each subgraph can be drawn independently with straight line segment edges
Vertices **need not have consistent positions** in all drawings





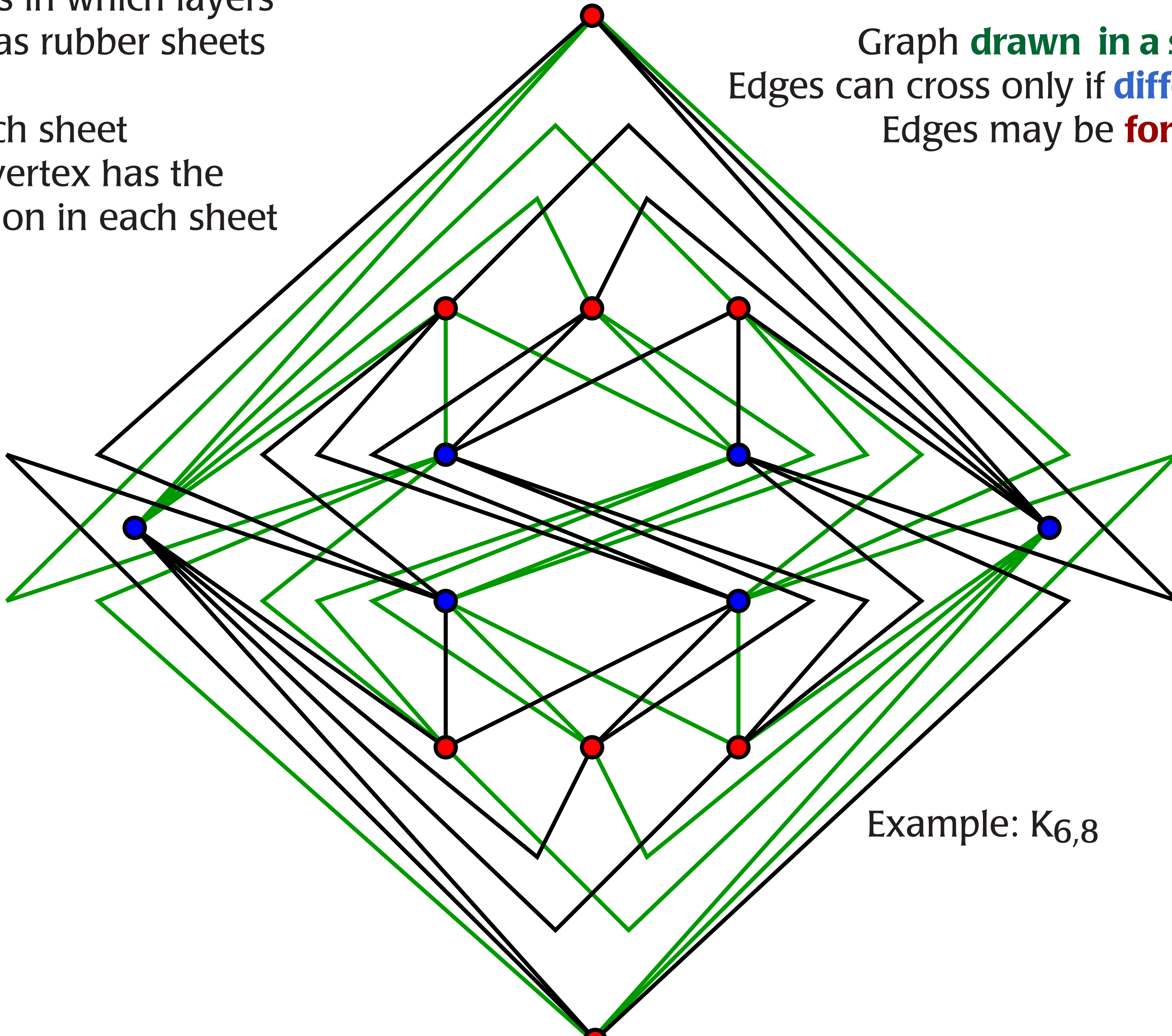
Example: $K_{6,8}$

Thickness definitions: graph thickness *(alternative def)*

View planes in which layers are drawn as rubber sheets

Deform each sheet until each vertex has the same location in each sheet

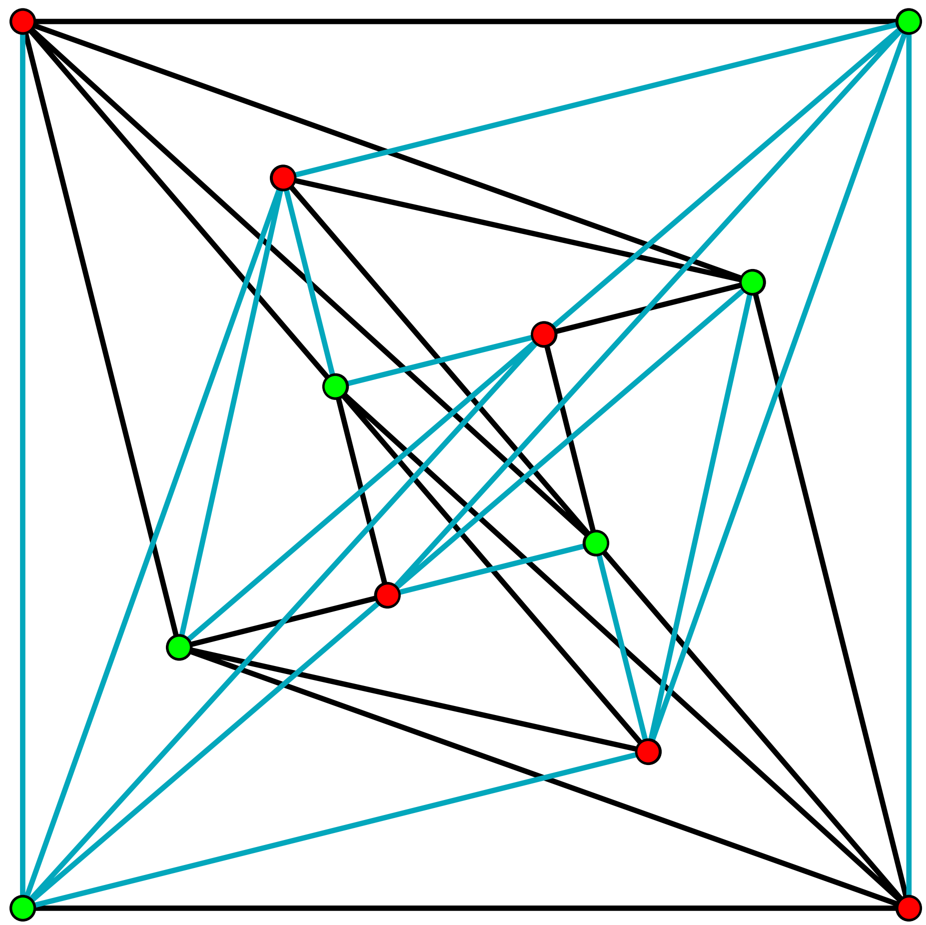
Result:
Graph **drawn in a single plane**
Edges can cross only if **different colors**
Edges may be **forced to bend**



Example: $K_{6,8}$

Thickness definitions: geometric thickness

Draw graph in the plane
each vertex as a **single point** (not restricted to convex position)
each edge as a **straight** line segment (not allowed to bend)
edges allowed to cross only when they have **different colors**



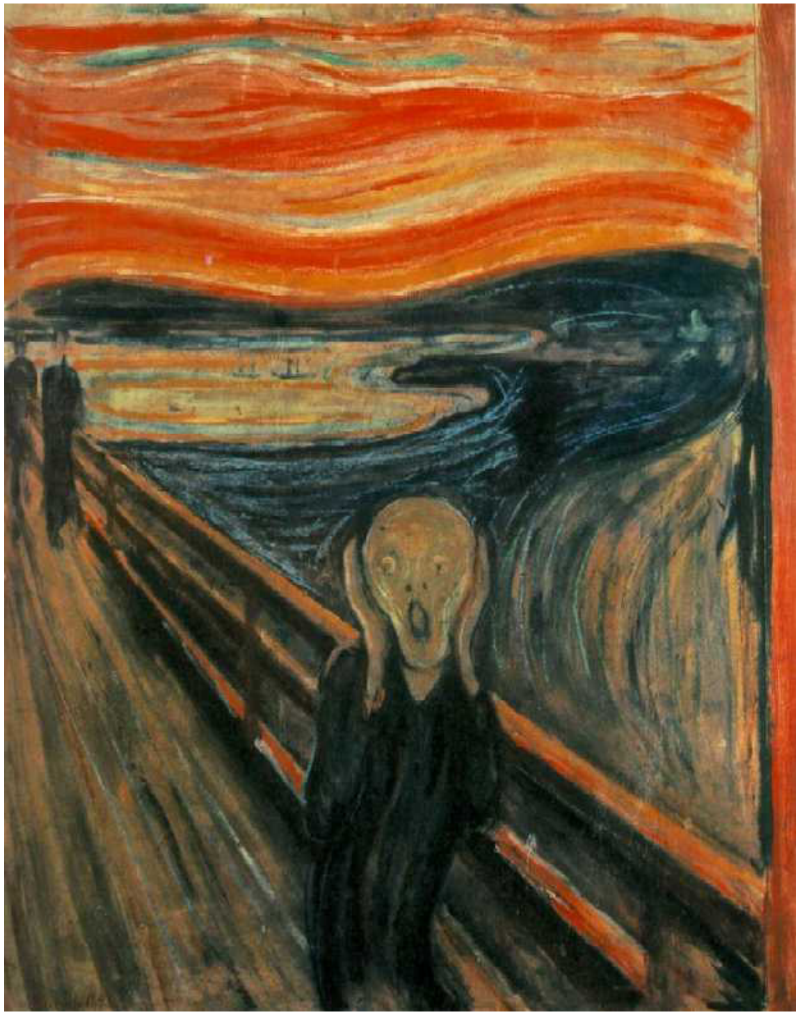
Example: $K_{6,6}$

Some complexity theory:

Book thickness: **NP-complete**
(Hamiltonian triangulation of planar graphs)

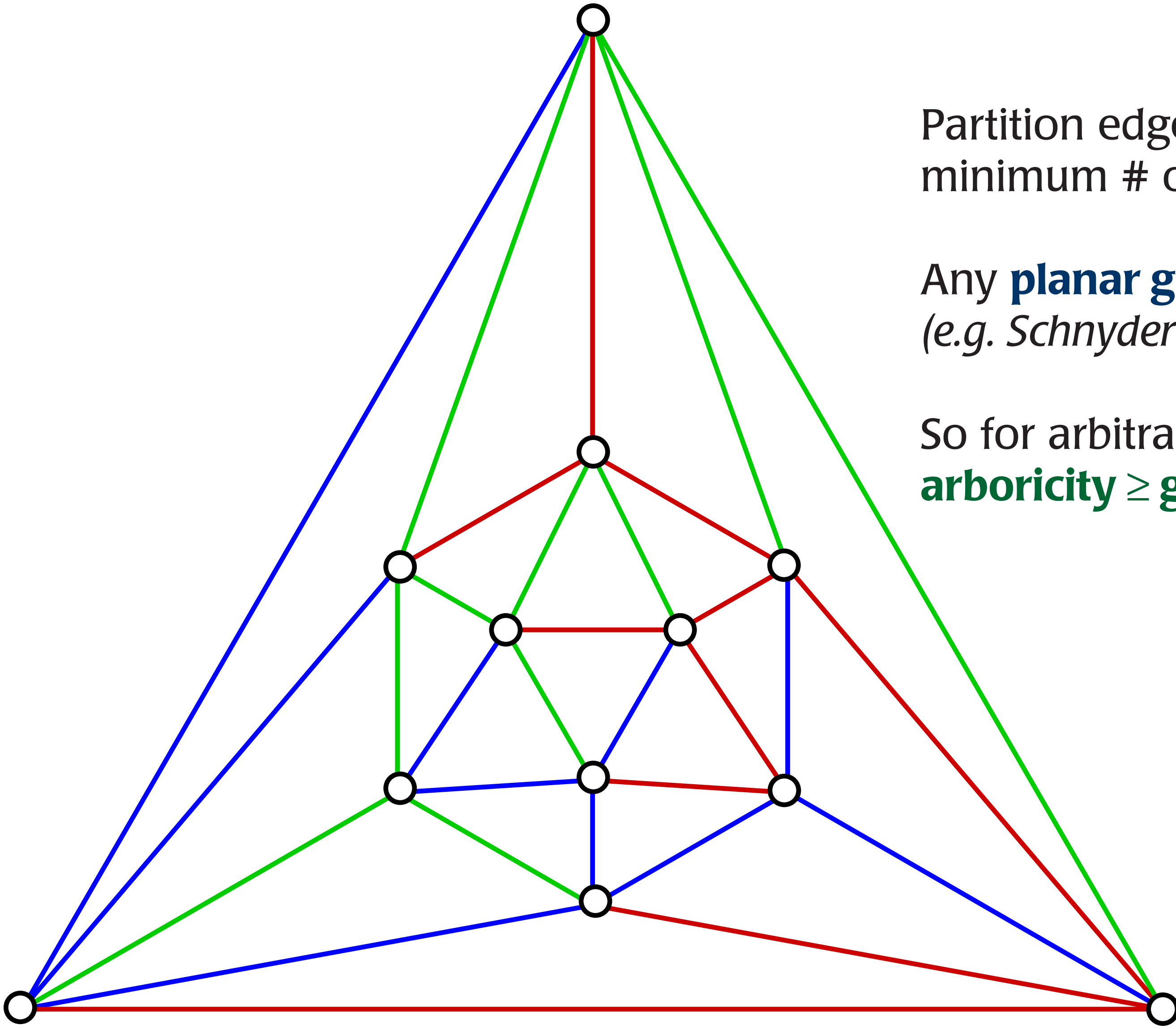
Geometric thickness: **Unknown complexity**
(likely to be NP-complete)

Graph thickness: **NP-complete**
(using rigidity of $K_{6,8}$ drawing)



So, if we can't quickly find optimal drawings of these types,
How well can we approximate the optimal number of colors?

Constant factor approximation to graph thickness:



Partition edges of graph into minimum # of forests (**arboricity**)

Any **planar graph** has arboricity ≤ 3
(e.g. Schnyder's grid drawing algorithm)

So for arbitrary graphs,
arboricity ≥ graph thickness ≥ arboricity/3

How similar are book thickness, geometric thickness, and graph thickness?

Equivalently, does arboricity approximate other kinds of thickness?

Known: book thickness $\geq$ geometric thickness $\geq$ graph thickness

Known: geometric thickness $\neq$ graph thickness

K6,8 has graph thickness two, geometric thickness three

More generally, ratio $\geq$ 1.0627 for complete graphs
[Dillencourt, Eppstein, Hirschberg, Graph Drawing '98]

Known: book thickness $\neq$ geometric thickness

Maximal planar graphs have geometric thickness one, book thickness two
More generally, ratio $\geq$ 2 for complete graphs

Unknown: how big can we make these ratios?

New results:

Ratio between book thickness and geometric thickness is not bounded by any constant factor

We describe a family of graphs $G_2(k)$
geometric thickness of $G_2(k) = 2$
book thickness of $G_2(k)$ is unbounded

Ratio between geometric thickness and graph thickness is not bounded by any constant factor

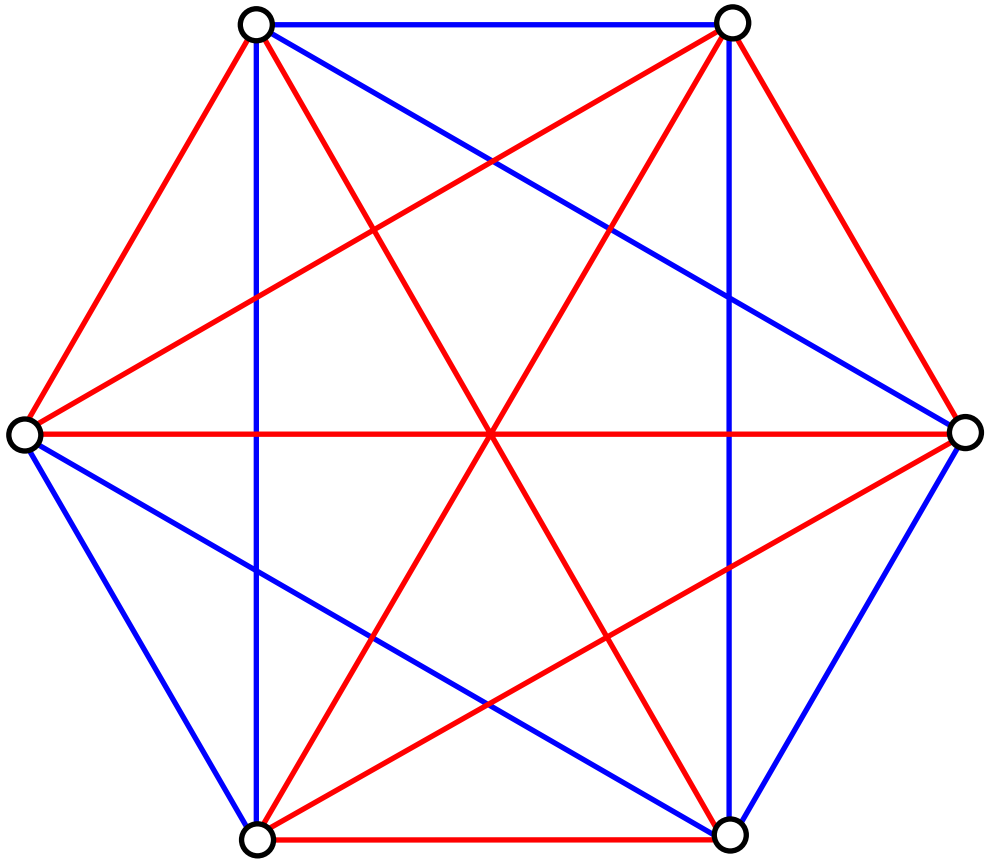
We describe a family of graphs $G_3(k)$
graph thickness of $G_3(k) \leq$ arboricity of $G_3(k) = 3$
geometric thickness of $G_3(k)$ is unbounded

Key idea: Build families of graphs by modifying complete graphs
Show directly that one kind of thickness is small
Use Ramsey Theory to amplify other kind of thickness

Ramsey Theory

General idea: large enough structures have highly ordered substructures

For any number of colors **r**, and number of vertices **k**,
we can find a (large) integer $\mathbf{R_r(k)}$
so that if we use **r** colors to color the edges of a complete graph on $\mathbf{R_r(k)}$ vertices
then some **k**-vertex complete subgraph uses only one edge color



Example: $R_2(3) = 6$

Ramsey Theory

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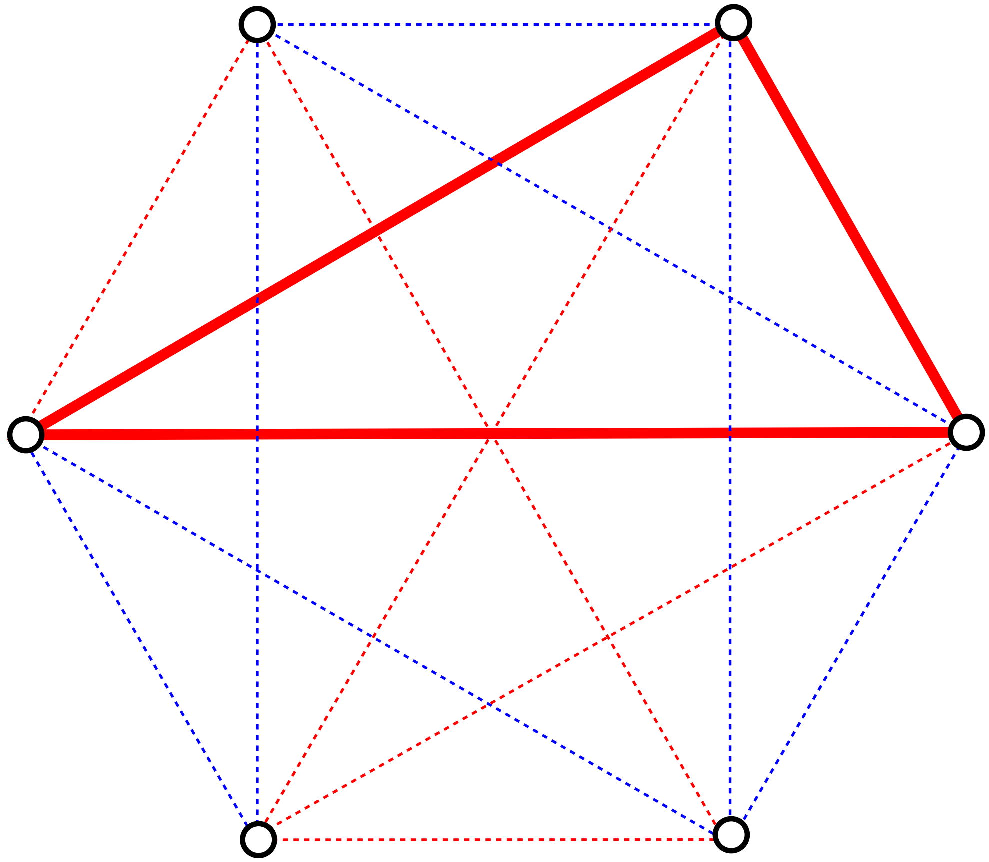


Example: $R_2(3) = 6$

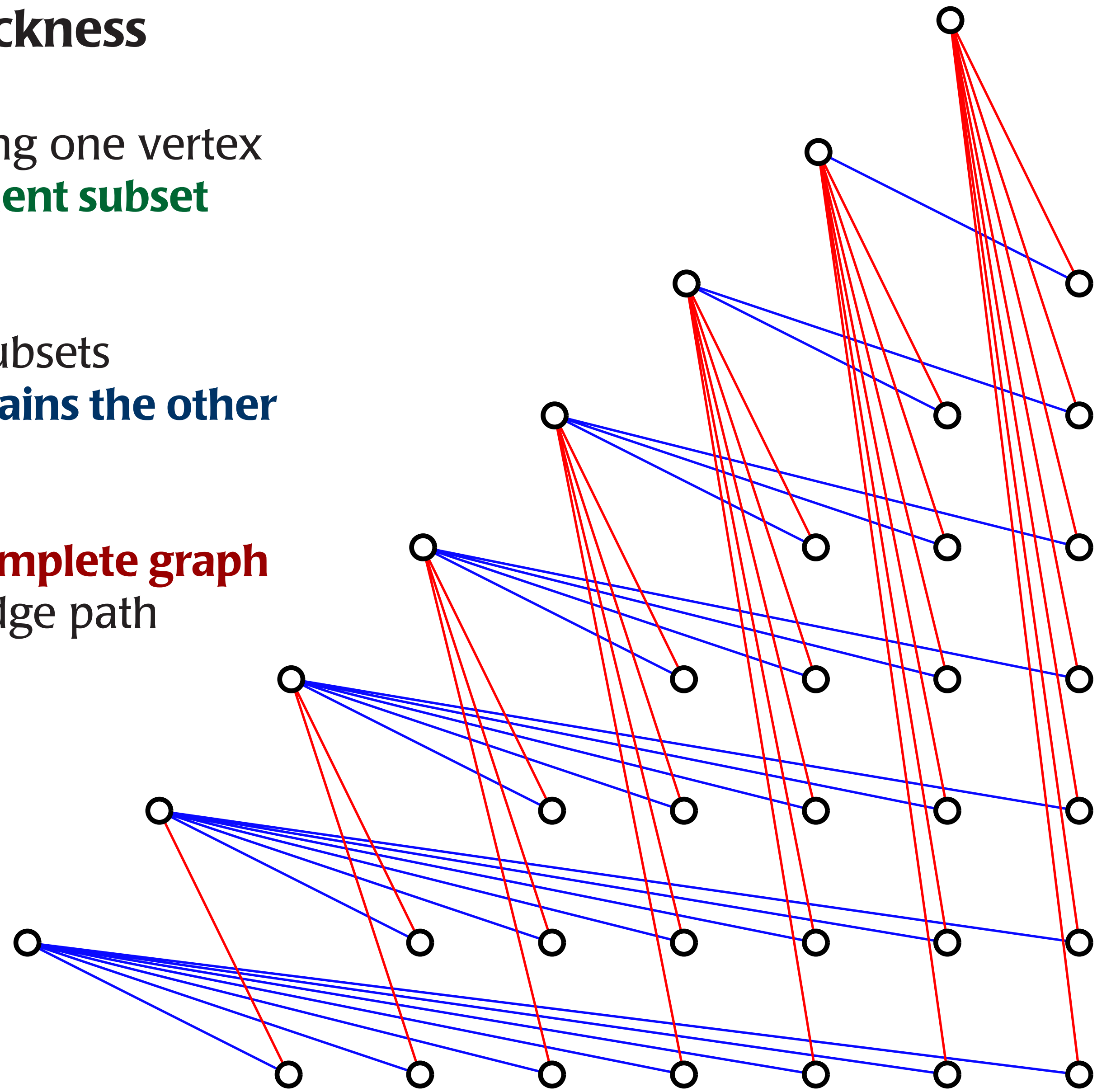
Graphs with high book thickness and low geometric thickness

Form graph $G_2(k)$ by making one vertex for **each one- or two-element subset** of a k-element set

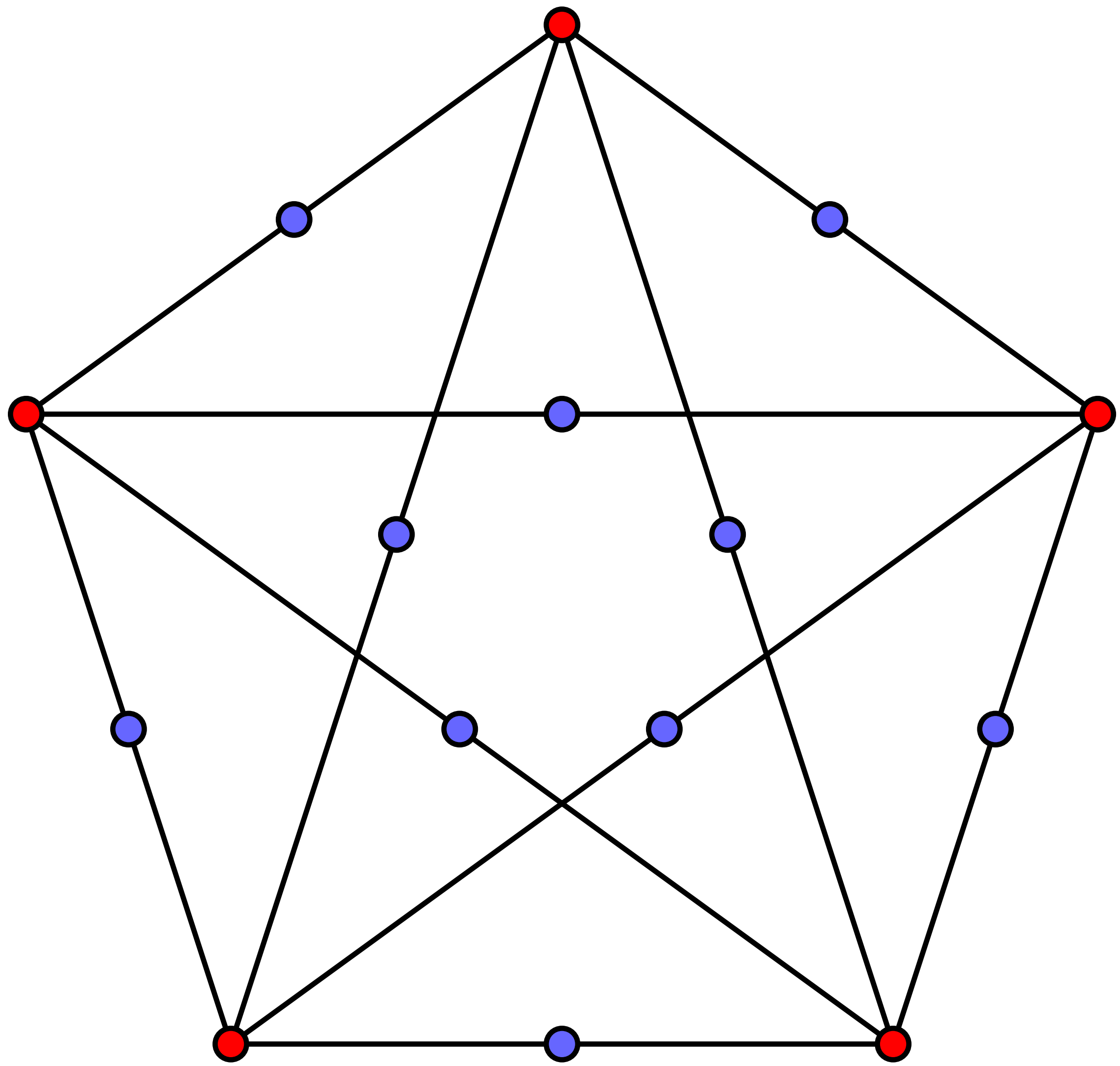
Place edge between two subsets whenever **one subset contains the other**

Equivalently...
subdivide each edge of complete graph on k vertices, into a two-edge path

$G_2(k)$ can be drawn with **geometric thickness = 2**



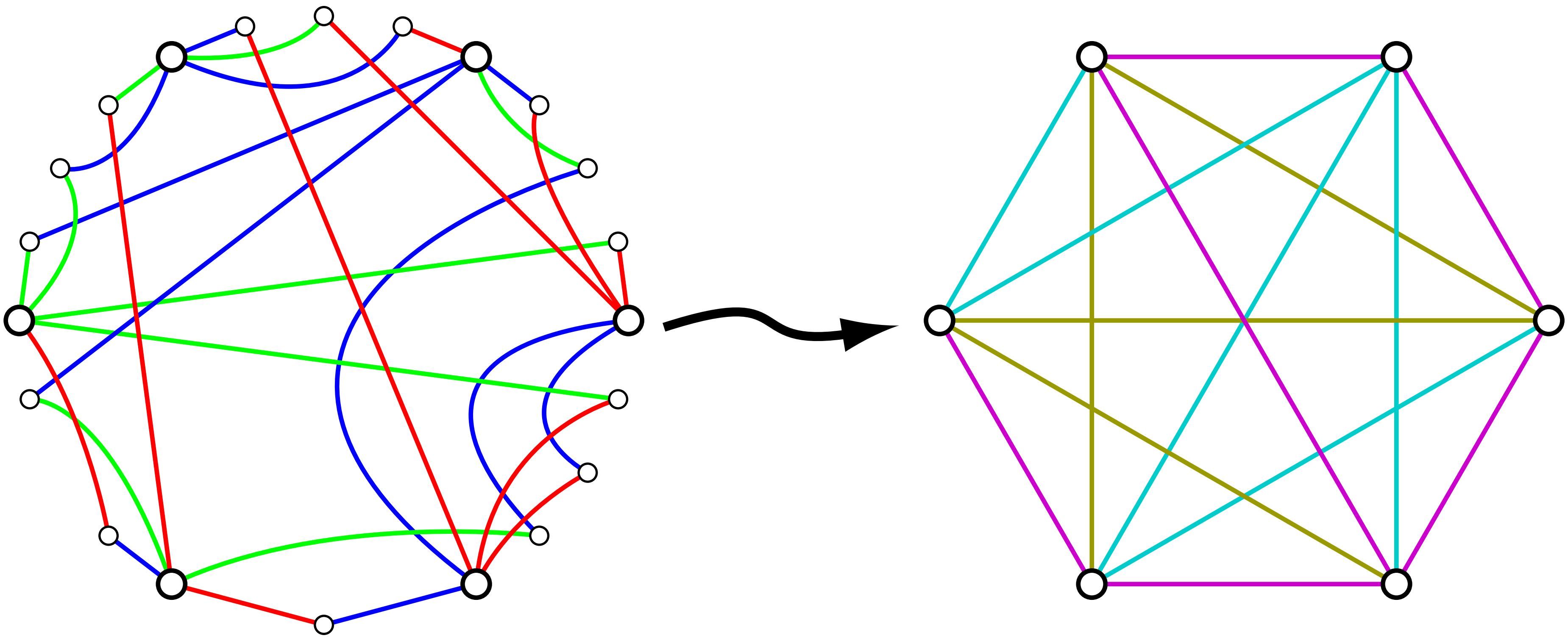
$G_2(5)$ has book thickness greater than two ...



... because it's not planar

Books with two leaves can be flattened into a single plane

For any t, $G_2(R_{t(t-1)/2}(5))$ has book thickness greater than t



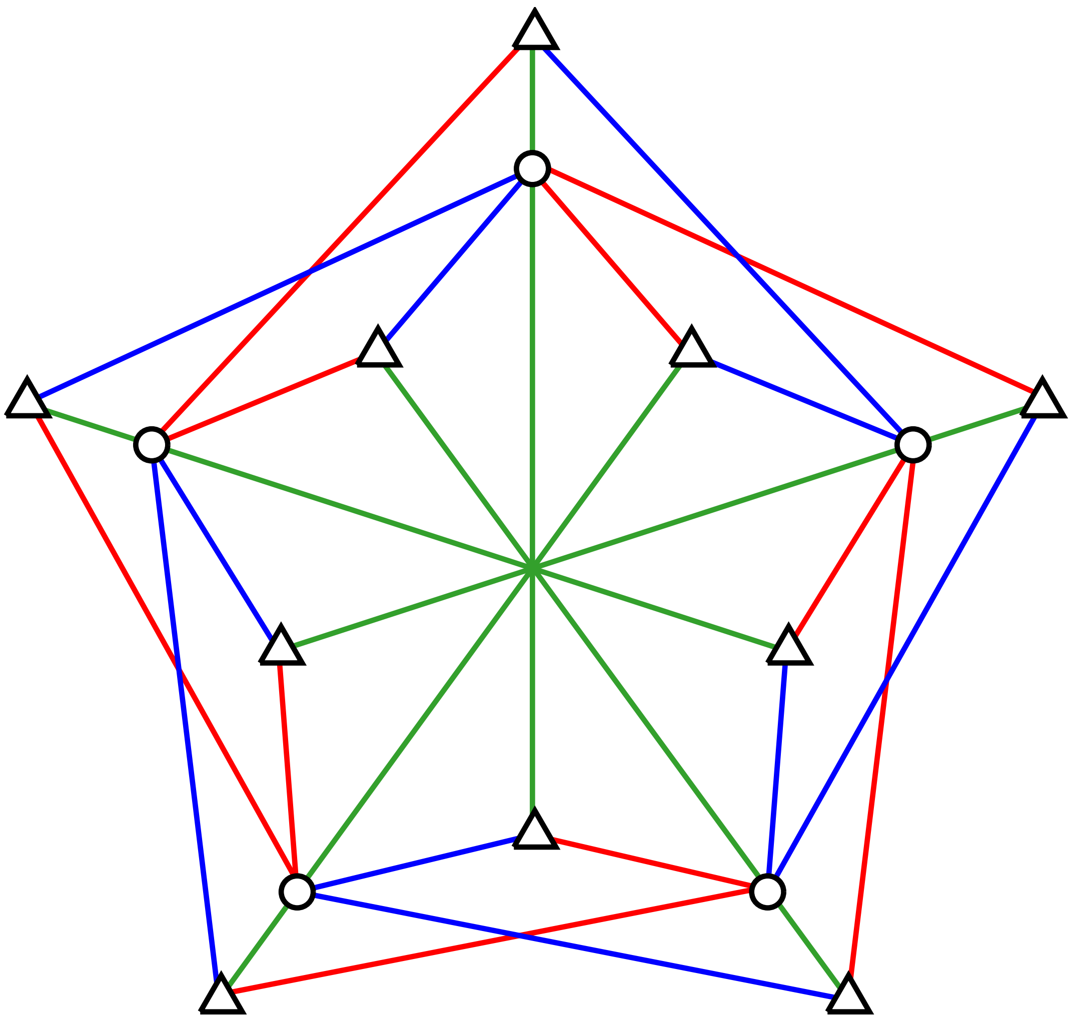
Proof:

Let $R = R_{t(t-1)/2}(5)$
Suppose we had a book drawing of $G_2(R)$ with only t layers
view pairs of layers in each 2-edge path as single colors of edges of K_R

Ramsey theory provides monochromatic K_5
corresponds to $G_2(5)$ subgraph of $G_2(R)$ with only two layers
But $G_2(5)$ does not have book thickness two, contradiction

Graphs with high geometric thickness and low book thickness

Form graph $G_3(k)$
vertices: one-element (**singleton**) and three-element (**tripleton**) subsets of k-element set
edges: **containment relations** between subsets



Arboricity ≤ 3:
Use different colors for the three edges at each tripleton vertex

Forms **forest of stars**

So graph thickness also ≤ 3.

Same Ramsey theory argument (with multicolored triples of complete hypergraph) shows geometric thickness unbounded

...as long as some $G_3(k)$ has geometric thickness > 3

The hard part: showing geometric thickness > 3 for some $G_3(k)$

Assume three-layer drawing given with sufficiently large k, prove in all cases crossing exists
Use Ramsey theory **[X]** repeatedly to simplify possible cases.

- Without loss of generality singletons form convex polygon **[X]**; number vertices clockwise
- Tripletons are either all inside or all outside the polygon **[X]**
- Layers given by low, middle, high numbered singleton incident to each tripleton **[X]**

- Case 1: Tripletons are all inside the convex polygon
  All tripletons can be assumed to cross the same way (convexly or concavely) **[X]**
    - Case 1A: all cross convexly
    - Case 1B: all cross concavely

  Either case forms grid of line segments forcing two tripletons to be out of position

- Case 2: Tripletons are all outside the convex polygon
  Classify tripletons by order in which it sees incident low, middle, high edges
  All tripletons can be assumed to have the same classification **[X]**
    - Case 2A: order high, low, middle. Can find two crossing tripletons.
    - Case 2B: order middle, high, low. Symmetric to 2A.
    - Case 2C: order low, high, middle. Convex polygon has too many sharp angles.
    - Case 2D: order middle, low, high. Symmetric to 2C.
    - Case 2E: order low, middle, high. All middle edges cross one polygon side;
      find smaller drawing where all low edges also cross same side; form grid like case 1.
    - Case 2F: order high, middle, low. All middle edges avoid the polygon;
      find smaller drawing where low edges also avoid polygon **[X]**; form grid.

Conclusions

All three concepts of thickness differ by non-constant factors

Arboricity is not a good approximation to geom. or book thickness

Open Problems

How big is the difference?
Our use of Ramsey theory leads to very small growth rates

What about other families of graphs?
Do bounded-degree graphs have bounded geometric thickness?

If graph thickness is two, how large can geometric thickness be?
We only prove unbounded for graph thickness three

Is optimizing geometric thickness NP-complete?

Can we efficiently find graph drawings with
nearly-optimal geometric thickness?

Combine thickness w/other drawing quality measures e.g. area?
What about other constraints on multilayer drawing e.g. O(1) bends?
Wood looked at one-bend drawing area but allowed edges to change color at the bend