Reflections in an Octagonal Mirror Maze

David Eppstein
University of California, Irvine

Canadian Conference on Computational Geometry, August 2022

Computational complexity of reflection

[Morin 2019]

To render a scene involving repeated mirror reflections, do we have to follow light rays one reflection at a time, or can we use shortcuts?

Finite systems can have many reflections

[Utamaro 1795]

Two parallel flat mirrors form paths with infinitely many reflections

But you can only see finitely many!

Your own head blocks the perpendicular (infinite) paths

All others are finite

From the continuous to the discrete

A drastic (over)simplification:

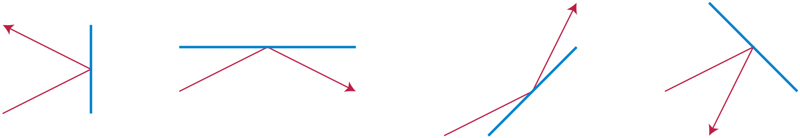
2D environment of axis-parallel and slope-± 1 mirrors and obstacles

All endpoints have integer coordinates

Viewpoint has integer coordinates and rational slope

Why these restrictions?

Reflected slopes are limited to a small set!



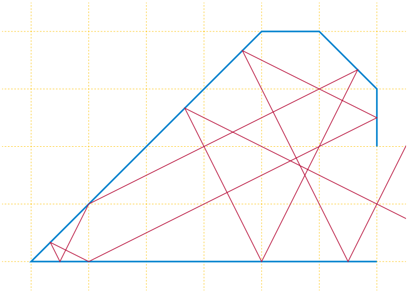
$$s \to \{\pm s, \pm \frac{1}{s}\}$$

... and the reflected ray stays on a line through integer points

Main new result

Given a 2D octagonal scene and a rational ray

We can find the eventual fate of the ray (the obstacle and angle that it hits, or a ray it escapes along) in polynomial time



Weakly polynomial: Depends on # bits of input coordinates, not just on # mirrors and obstacles in input scene

But number of reflections can be exponential in the same quantity, so this is much better than step-by-step simulation

Main ideas

Transform into a different problem!

Mirror reflection problem

⇒ Partial interval exchange

⇒ Interval exchange transformation

⇒ Normal curves on topological surfaces

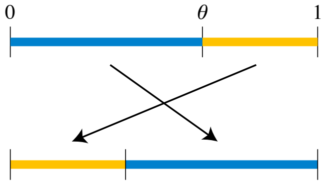
⇒ Known algorithms

[prayitno 2012]

What is...an interval exchange transformation?

Partition an interval into subintervals and permute them

$\Rightarrow$ piecewise-translational bijection



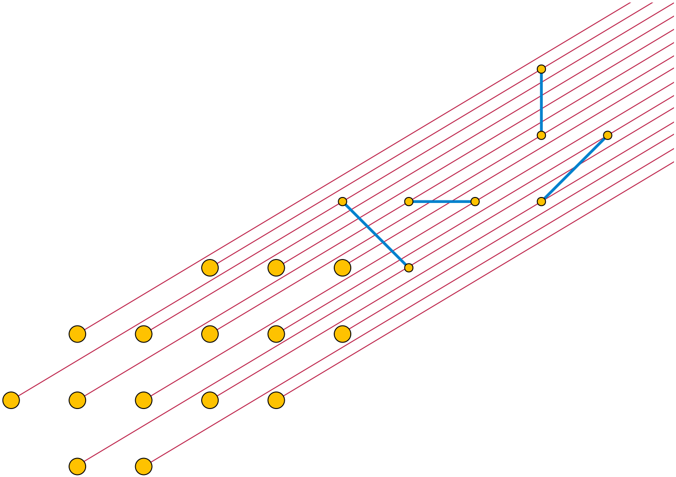
We are interested in intervals of *integers*
and integer-preserving transformations of them:

"Integer interval exchange transformation"

Description size = # subintervals $\times$ $\log_2$(interval length)

How fixed-slope integer-origin rays hit a mirror

Points of intersection are non-integer but evenly spaced

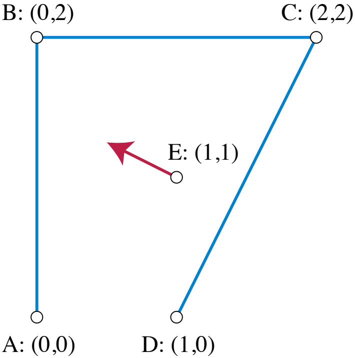


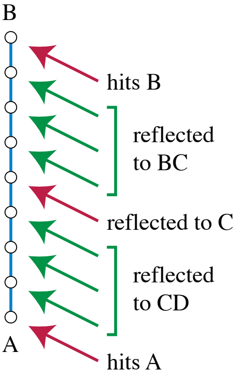
We can compute how many from the slope of the ray and mirror

From mirrors to interval exchange

For each mirror and each of $O(1)$ possible incident slopes:

- Count positions at which a ray of that slope can hit
- Concatenate all these sequences of positions
  for (mirror, slope) pairs into one big integer interval
- Form subintervals for where a reflected ray will go next

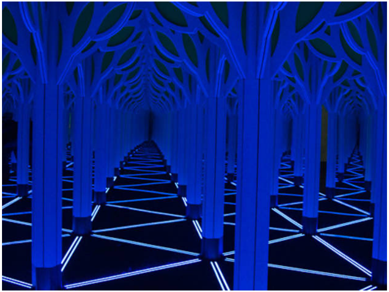




But where do the blocked/escaping rays go?

An interval exchange transformation must be a bijection

Make a “trap”: part of the transformation that maps long subinterval to itself + small shift (like two parallel mirrors) so it takes huge number of steps to escape



[Kevdog686 2019]

Transform rays with no next reflection to start of trap

Transform rays with no preimage from end of trap

Transformed problem, so far

Turn input scene and line of sight into an
equivalent integer interval exchange transformation τ

- # intervals = quadratic in # scene features
- length of interval = polynomial in scene coordinates
- initial ray = initial value x for transformation
- every ray is trapped in $\leq N$ steps, for polynomial N
- trap takes more than N steps to escape

$\Rightarrow$ We just need to compute $\tau^{(N)}(x)$ and decode it!

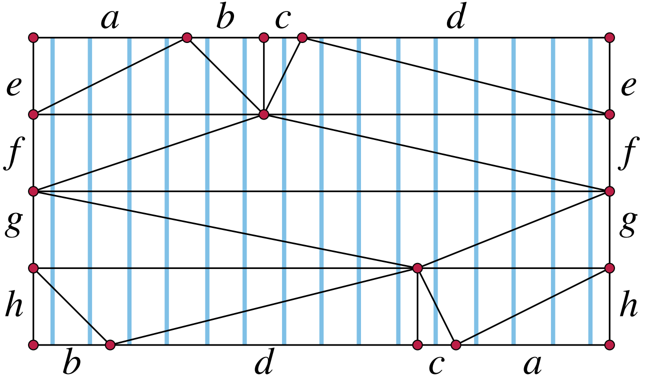
From interval exchange to topology

Triangulated rectangle + glued boundary + vertical "normal curves"

Top: intervals of exchange transformation

Bottom: their permutation

Middle: single edge of triangulation

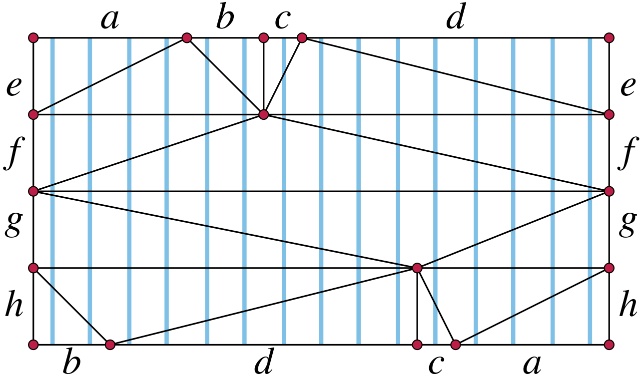


$\tau^{(N)}(x)$: start at position x on middle edge, then travel vertically for height $\times\, 2N$ steps along normal curve

The algorithm

Transform mirror scene into exchange transformation

Transform transformation into normal curves of triangulated surface



Compute what happens if you travel k steps along a normal curve using algorithms from [Erickson and Nayyeri 2013]

Translate everything back into terms of original problem

A related open problem

Suppose a convex polygon is made of a highly-refractive material

Light ray enters some edge at angle $\geq \theta$
(for threshold θ depending on index of refraction)
bounces around internally at angles $< \theta$
and exits when it hits another edge at angle $\geq \theta$

Can we find exit point without simulating bounce-by-bounce?

[Fatimazt 2018]

References and image credits, I

Jeff Erickson and Amir Nayyeri. Tracing compressed curves in triangulated surfaces. *Discrete & Computational Geometry*, 49 (4):823–863, 2013. doi: 10.1007/s00454-013-9515-z.

Fatimazt. Daimond de mer. CC-BY-SA image, August 3 2018. URL `https://commons.wikimedia.org/wiki/File:Daimond.jpg`.

Kevdog686. A picture of the inside of the mirror maze showcasing its infinite looking repetition. CC-BY-SA image, May 27 2019. URL `https://commons.wikimedia.org/wiki/File:Mirror_Maze_in_the_Museum_of_Science%2BIndustry_of_Chicago.jpg`.

Basile Morin. Street crowd reflecting in the polyhedral mirrors of the station Tokyu Plaza Omotesando, Harajuku, Tokyo, Japan. CC-BY-SA image, June 16 2019. URL `https://commons.wikimedia.org/wiki/File:Street_crowd_reflecting_in_the_polyhedral_mirrors_of_the_station_Tokyu_Plaza_Omotesando,_Harajuku,_Tokyo,_Japan.jpg`.

References and image credits, II

prayitno. Optimus Prime & Bumble Bee TRANSFORMERS THE RIDE-3D. CC-BY image, May 20 2012. URL https://commons.wikimedia.org/wiki/File:Transformers_costume_characters_at_Universal_Studios_Hollywood.jpg.

Kitagawa Utamaro. Takashima Ohisa Using Two Mirrors to Observe Her Coiffure. Public-domain image, 1795.