Topological Issues in Hexahedral Meshing

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Outline

I. What is meshing?

Problem statement – Types of mesh – Quality issues – Duality

II. What can we mesh?

Necessary conditions – Sufficient conditions for topological mesh –
Geometric existence problem – bicuboid

III. How well can we mesh?

Mesh complexity – Provable quality

IV. How can we make our meshes better?

Point placement – topological changes –
flipping – flip graph connectivity – bicuboid revisited

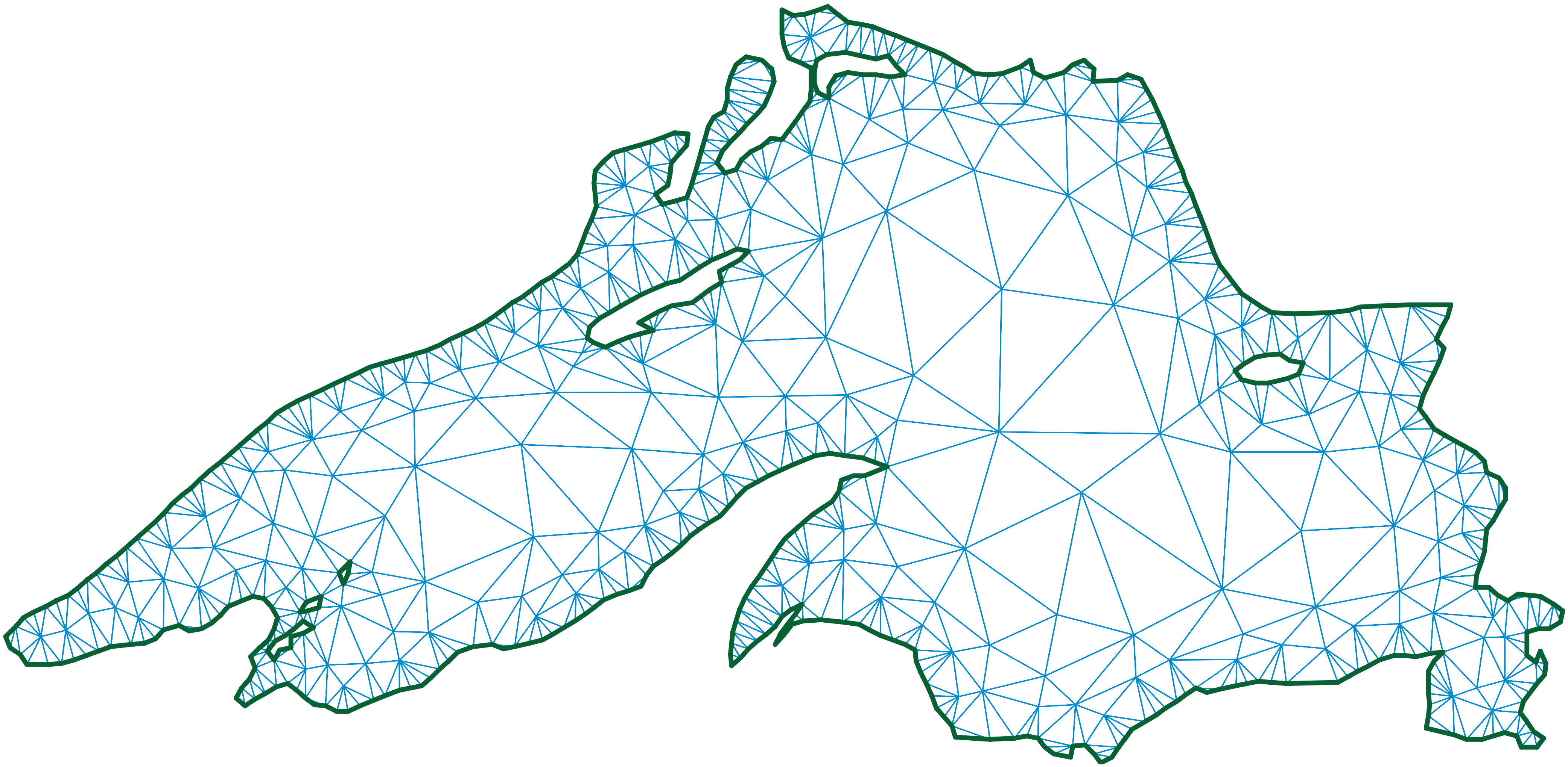
I. What is Meshing?

Given an input domain
(manifold with boundary or possibly non-manifold geometry)

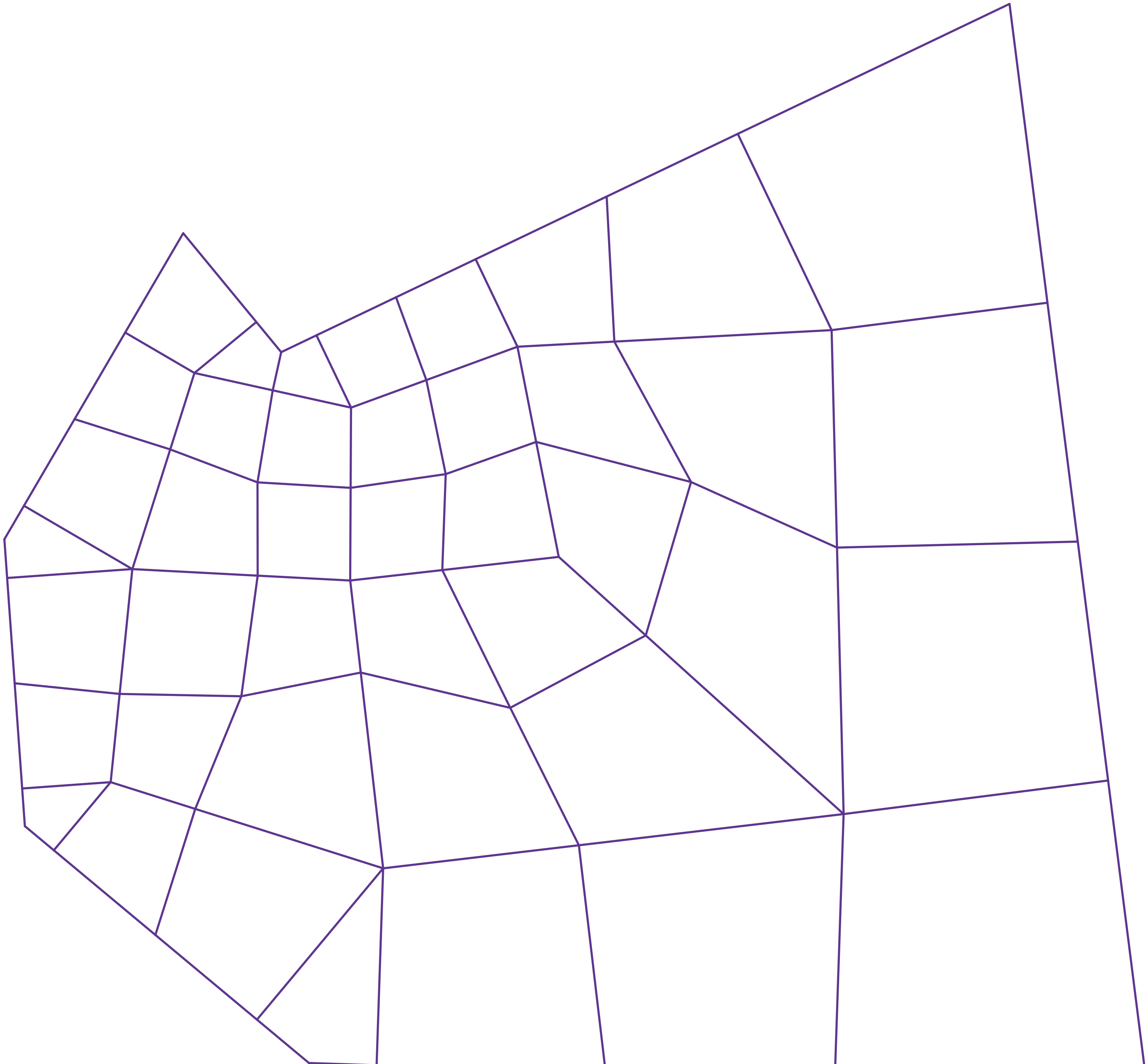
Partition it into simple cells
(triangles, quadrilaterals, tetrahedra, cuboids)

Essential preprocessing step for finite element method
(numerical solution of differential equations e.g. airflow)

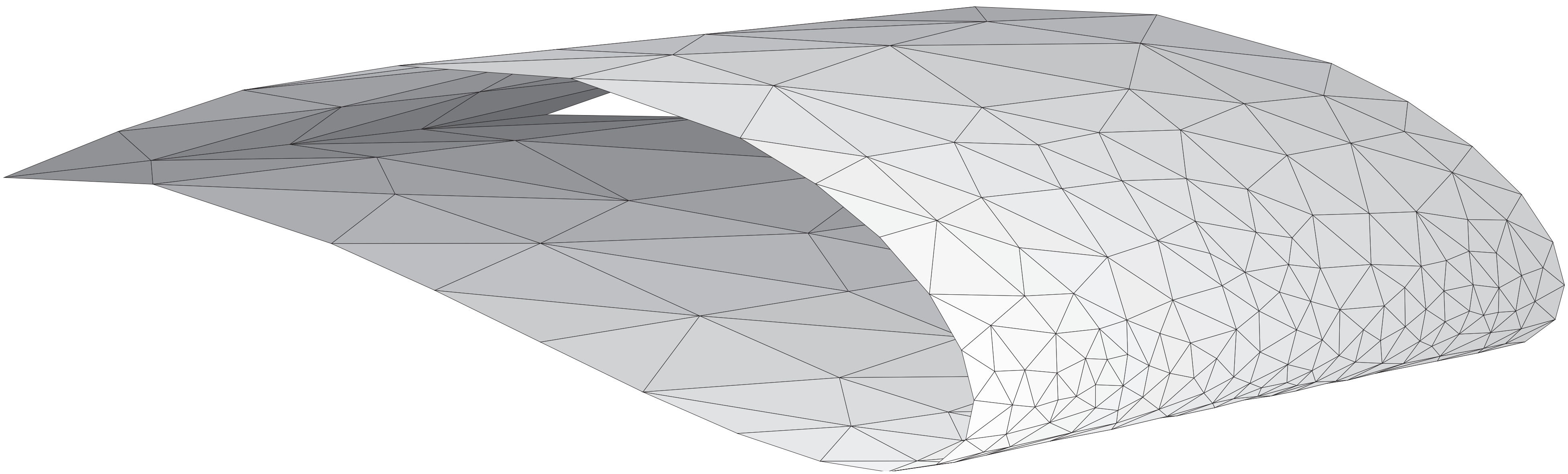
Other applications e.g. computer graphics



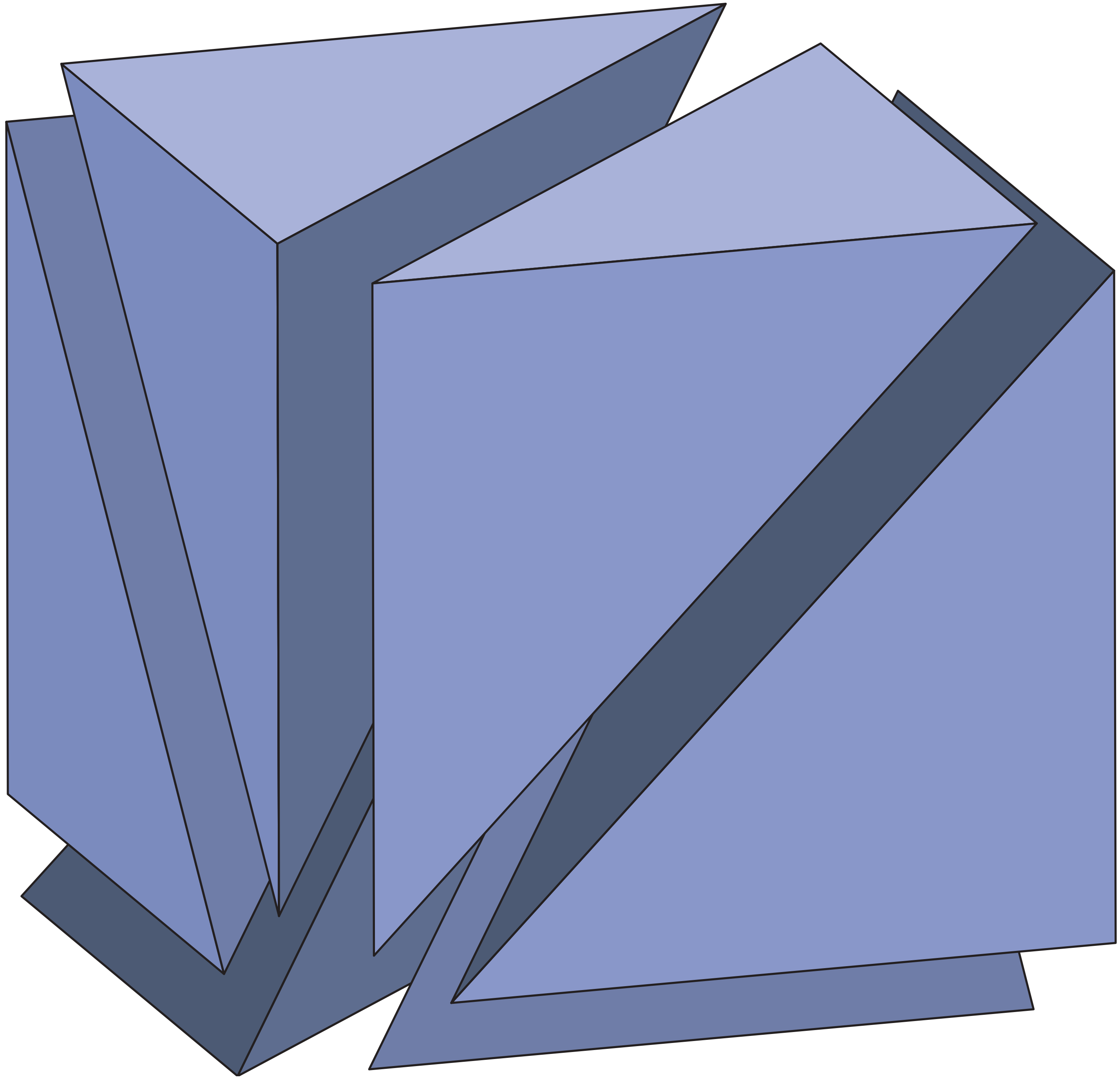
Triangle mesh of Lake Superior [Ruppert]



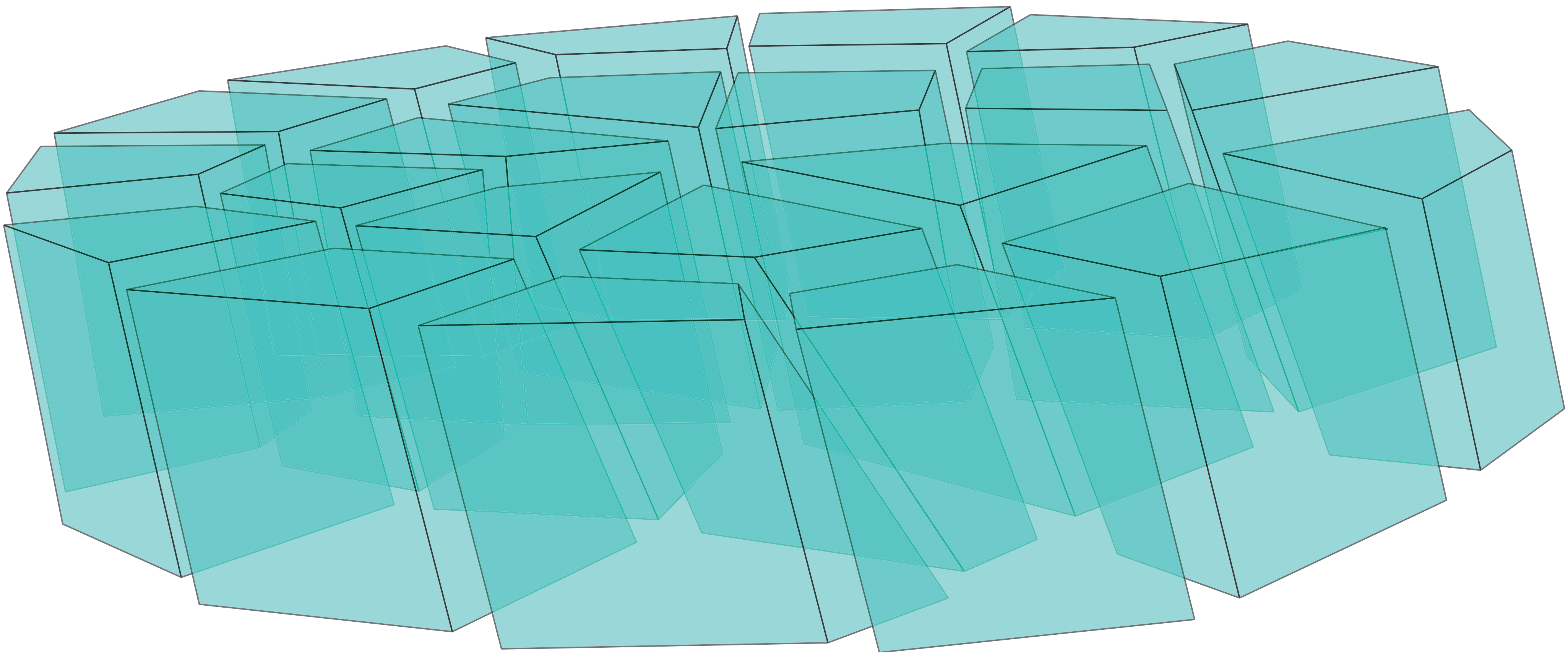
Quadrilateral mesh of an irregular polygon
(all quadrilaterals kite-shaped)



Triangle mesh on three-dimensional surface [Chew]



Tetrahedral mesh of a cube



Portion of hexahedral mesh of elbow pipe
[Tautges and Mitchell]

Mesh Quality Issues

Element type?

This talk: quadrilateral and hexahedral meshes

Element shape?

Avoid sharp angles, flat angles, distorted elements
Affects accuracy of numerical simulation

Element size?

Need small elements near small features or abrupt changes in solution
large elements ok in uninteresting parts of domain

Number of elements?

More elements = slower solution time

Elements on domain boundaries?

May be required to match existing domain boundary mesh
for quality reasons or to mesh multi-domain input

Why hexahedral Meshes?

Fewer elements

Fit man-made objects better

Better numerical behavior in some problems
e.g. stress analysis

Multiblock methods:
subdivide coarse hexahedra into fine regular cubical meshes

Less well understood = more interesting problems

Types of hexahedral mesh

Topological – abstract cell complex

Main focus of work on mesh existence
Not much use in practice

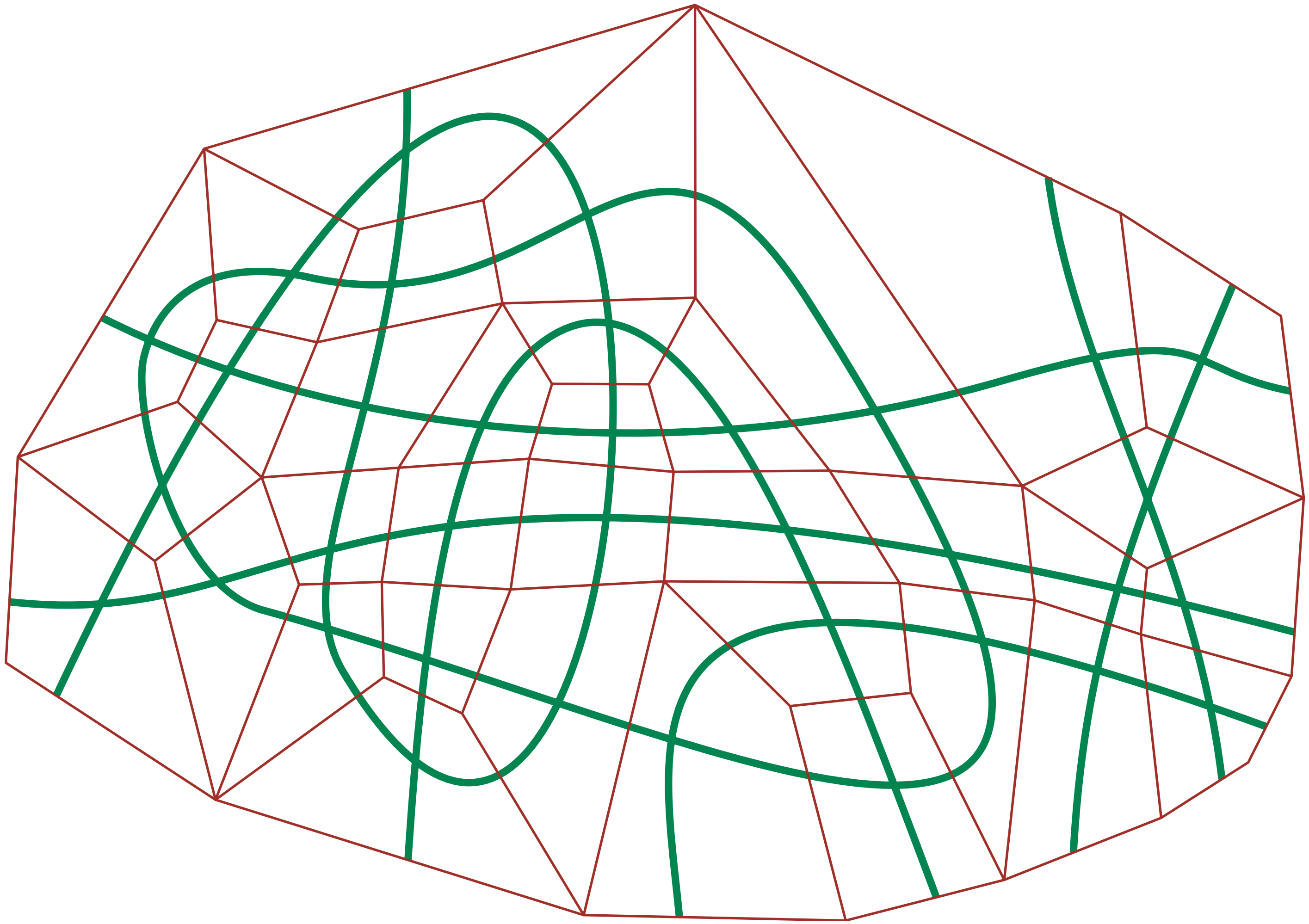
Warped – complex w/vertex locations

Cell facets are reguli bounded by warped quadrilaterals
Preferably non-self-intersecting
Main type of mesh used in practice

Geometric – polyhedral subdivision

Main focus of work in computational geometry
Not used much in practice because difficult or impossible to generate,
other quality criteria more important than flat facets

Duality for Quadrilateral Meshes

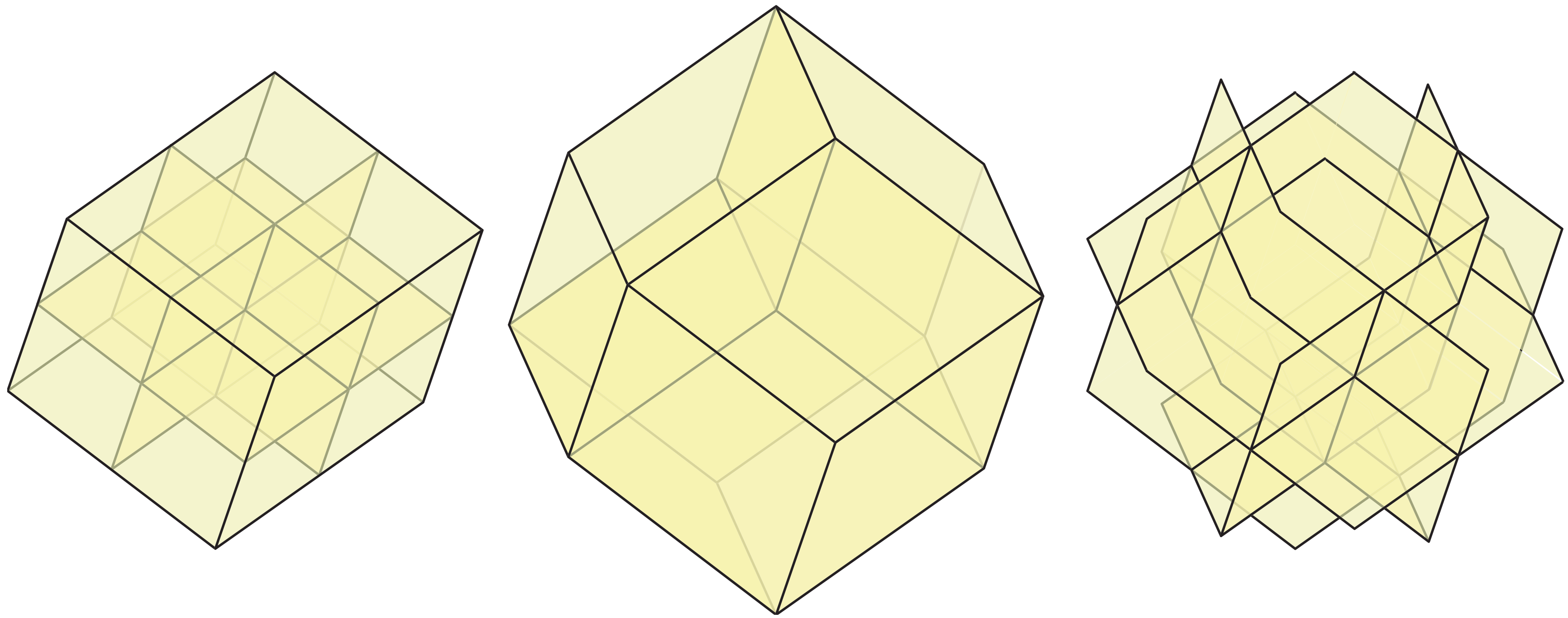


Draw curves connecting opposite edges of each quadrilateral
Subdivides quadrilateral into four pieces

Mesh corresponds to curve arrangement connecting midpoints of boundary edges
(connected, with no multiple adjacencies among arrangement vertices)

May possibly include curves nonadjacent to boundary

Duality for hexahedral meshes



Left: cuboid subdivided by three surfaces into eight pieces
Center: four-cuboid mesh of rhombic dodecahedron
Right: dual surface arrangement

Hex mesh corresponds to arrangement of surfaces
meeting domain boundary in dual of boundary quad mesh
connected skeleton, no pinch points, no multiple adjacencies

May possibly include surfaces nonadjacent to boundary
surface can self-intersect, no requirement of orientability

II. What can we mesh?

Simple necessary condition:
even number of domain boundary facets

E.g. for quadrilateral mesh, even number of edges

Why? Because each quad or hex has an even number,
and internal facets cancel in pairs

Sufficient for quadrilateral meshes

Choose points at each edge midpoint,
form curves connecting pairs of points

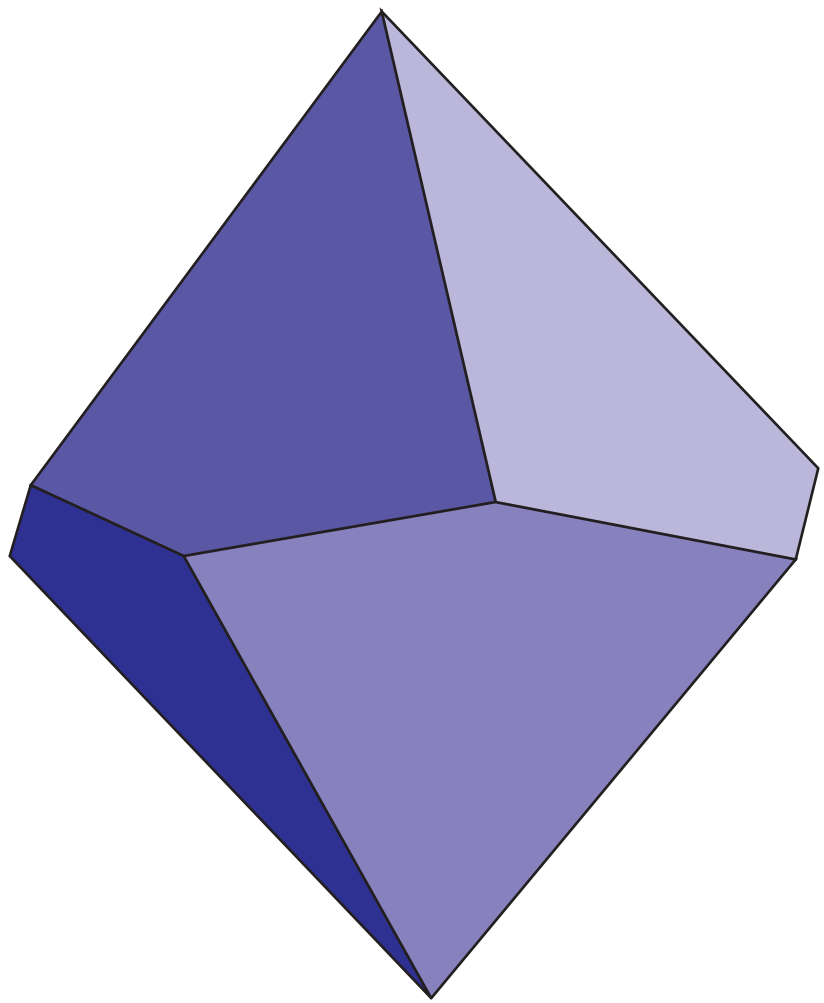
Use duality to turn curve arrangement into topological mesh

More complicated techniques can be used
to construct a geometric mesh of convex quadrilaterals

Hexahedral mesh existence?

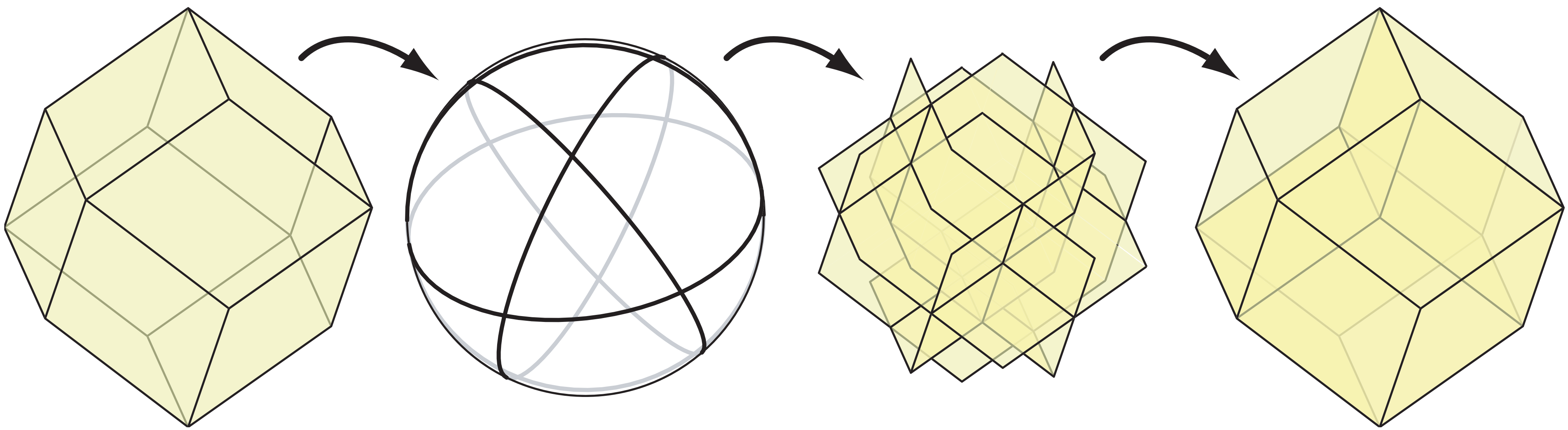
Given polyhedron with even number of quadrilateral facets,
when can we find a hexahedral mesh conforming to the boundary?

Example: the difficult octahedron



Mitchell and Thurston [1996] results:

Even # facets sufficient for topological mesh of simply connected 3d domains



Dualize boundary mesh to curve arrangement on sphere

Extend curves with even # self-intersections to surfaces [Smale]
pair up odd curves and similarly extend to surfaces

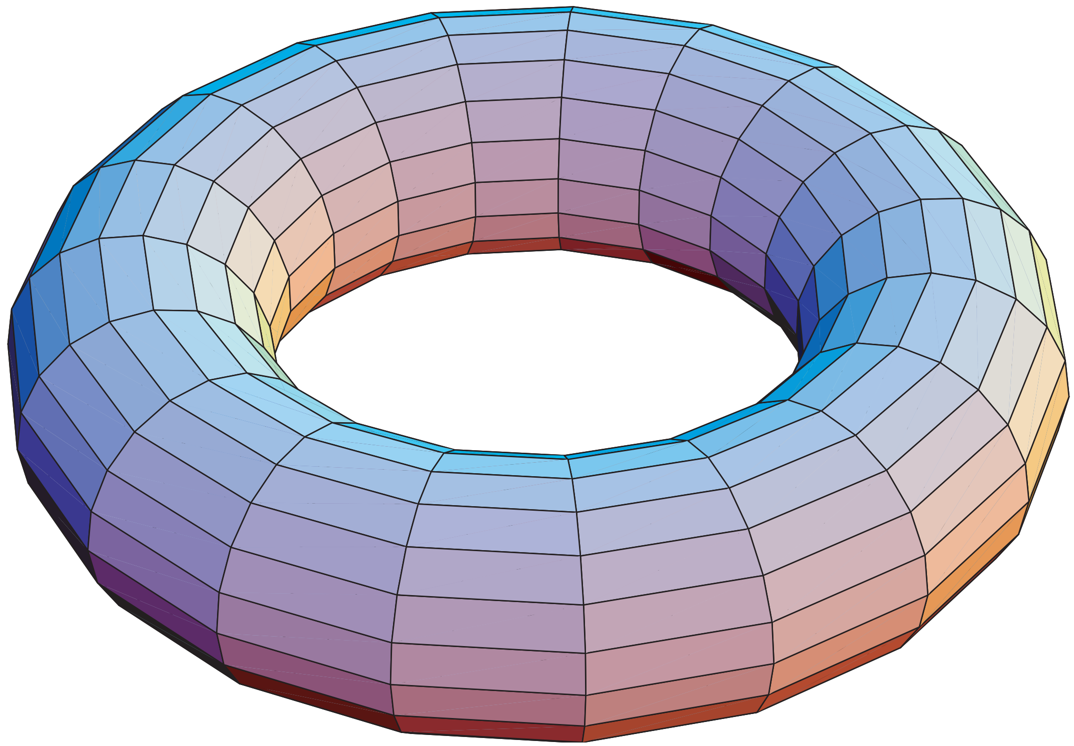
Add extra surfaces to enforce no-multiple-adjacency rules

Dualize surfaces back to hexahedral mesh

Extensions to non-simply-connected domains? [Mitchell & Thurston]

Necessary: no odd cycle of skeleton bounds a surface in domain

Because intersection with mesh's dual surfaces
would form curves with an even number of endpoints



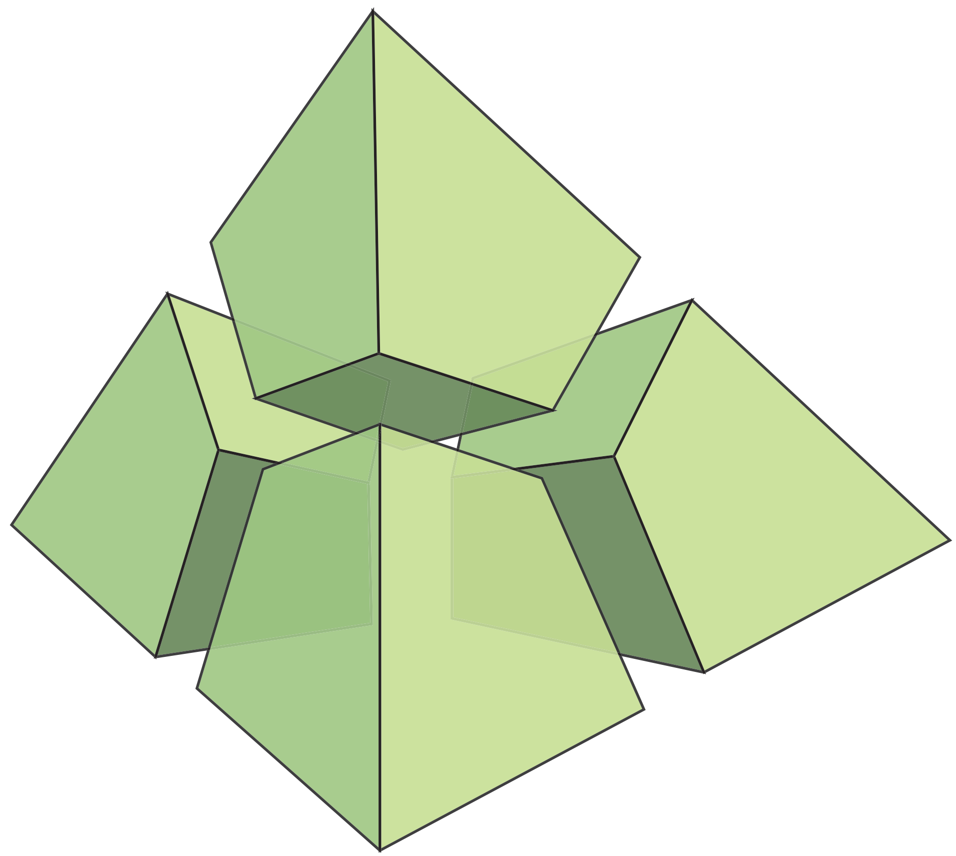
Sufficient: handlebody, each handle can be cut by an even cycle

Cut the handles by disks
Form quad mesh on each disk
Mesh the resulting simply-connected domain

Further extensions [Eppstein, 1996]

Sufficient: skeleton has no odd cycles (i.e. forms bipartite graph)

Separate domain boundary from interior by a layer of cuboids
Tetrahedralize interior
Subdivide each tetrahedron into four cuboids



Subdivide quadrilaterals between pairs of boundary cuboids to make all even
Re-mesh boundary cuboids via Mitchell & Thurston

Open questions for mesh existence:

Characterize which 3-manifolds have boundary-conforming topological hexahedral meshes

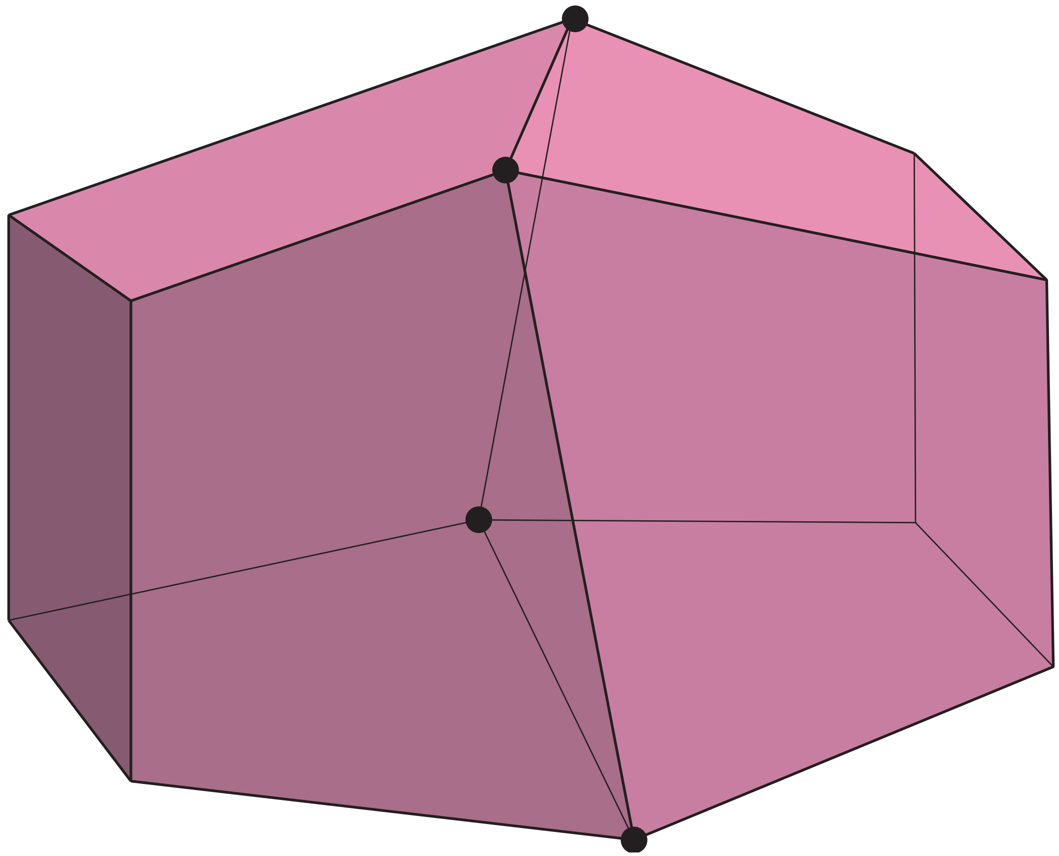
What is the simplest manifold where mesh existence still unknown?
Odd-by-even grids on knot complements?

Maybe handlebody technique extends to all Haken manifolds?

Open questions (continued):

Understand geometric hex mesh existence

E.g. does bicuboid with warped equator have a mesh?



Seems to be the hard case for geometric hex meshing more generally
(other domains can be decomposed into bicuboids)

III. How well can we mesh?

"Guaranteed quality": proof that results will always meet criterion

avoids problems with rare special cases, judgement "by eye"
still needs decision on what is the right criterion

Simplest criterion: number of elements

Triangle meshes: trivial to optimize

Tetrahedron meshes: NP-complete to optimize,
even for convex polyhedra [Below, De Loera, Richter-Gebert]

Quadrilateral meshes: can optimize for convex polyhedra,
NP-complete but approximable for domains with internal boundaries
[Müller-Hannemann and Weihe]

Hexahedral meshes: complexity of optimization open,
upper and lower bounds on numbers of elements known

Hexahedral meshing: bounds on numbers of elements

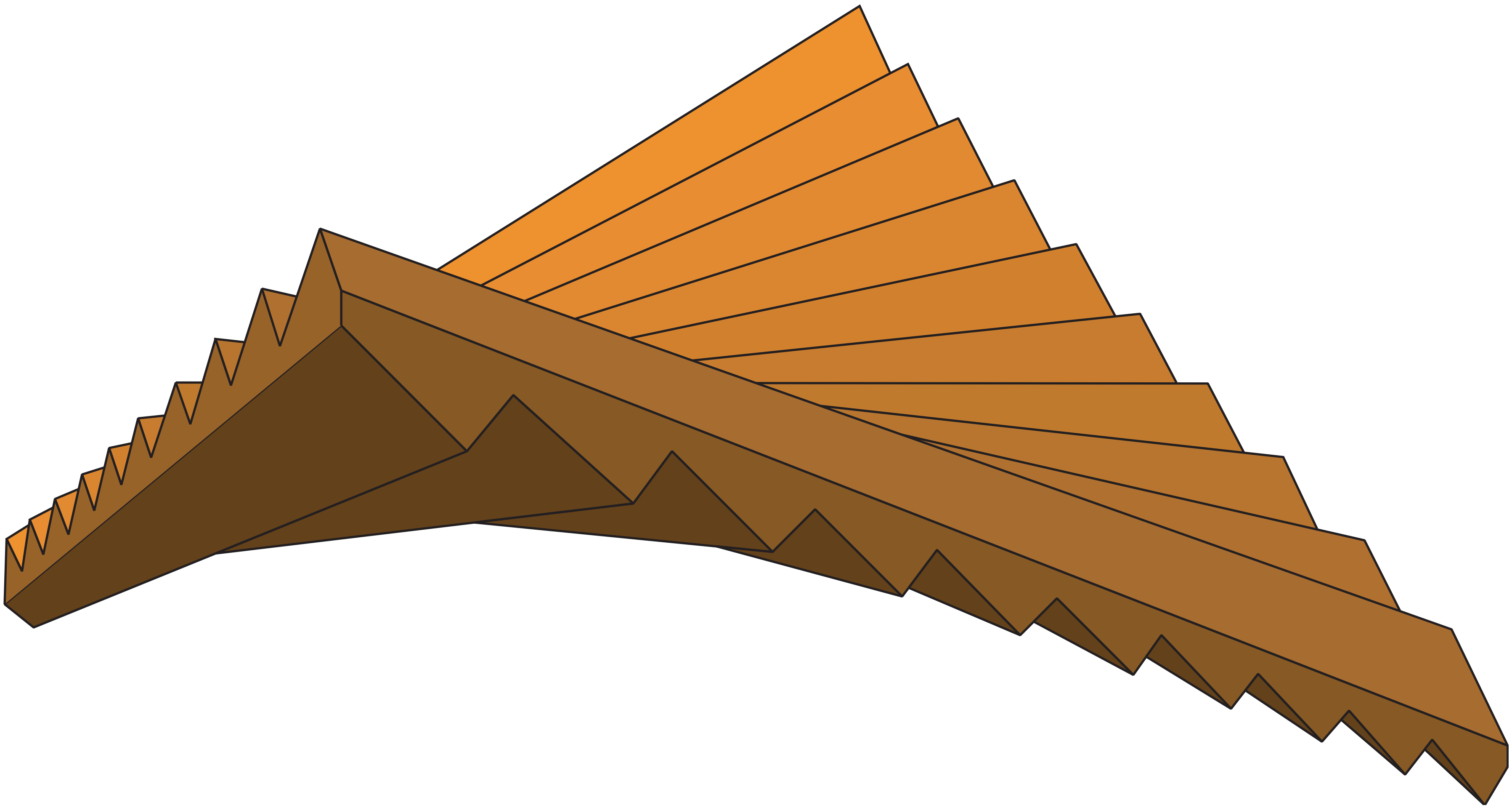
Mitchell-Thurston method:

$\Omega(n^2)$ with bad choice of pairing odd curves, $\Omega(n^{3/2})$ in general

Eppstein method:

$O(n)$ for topological mesh, might lead to $O(n^2)$ for geometric mesh

Some domains require $\Omega(n^2)$ for geometric mesh [Chazelle]



Combinations of shape and complexity criteria

Triangular meshes

Angles bounded away from zero, optimal # triangles [Bern, Eppstein, Gilbert]
Angles at most 90, linear # triangles [Bern, Mitchell, Ruppert]

Tetrahedral meshes

Angles bounded away from zero, optimal # tetras [Mitchell, Vavasis]

Quadrilateral meshes

Angles bounded below 120, linear # quads [Bern, Eppstein]

Angles bounded away from zero, bounded aspect ratio
only known solution subdivides triangle mesh

Hexahedral meshes

Little is known

IV. How can we make our meshes better?

If we don't know how to guarantee quality,
maybe we can still get meshes that are mostly good
by starting with not-so-good meshes and improving them

Even if we can guarantee quality,
often we can still find further improvements to make

Laplacian smoothing

Move all interior vertices to average of neighbor's positions
ad-hoc, does not preserve mesh topology

Optimization based smoothing

Move points one at a time in one or more passes over mesh
Place point to optimize quality measure of neighboring cells
Can allow a priori (shape based) or a posteriori (solution based) measures

Efficient placement possible using LP-type methods [Amenta, Bern, Eppstein]

Topology can be added constraint or automatic result of optimization

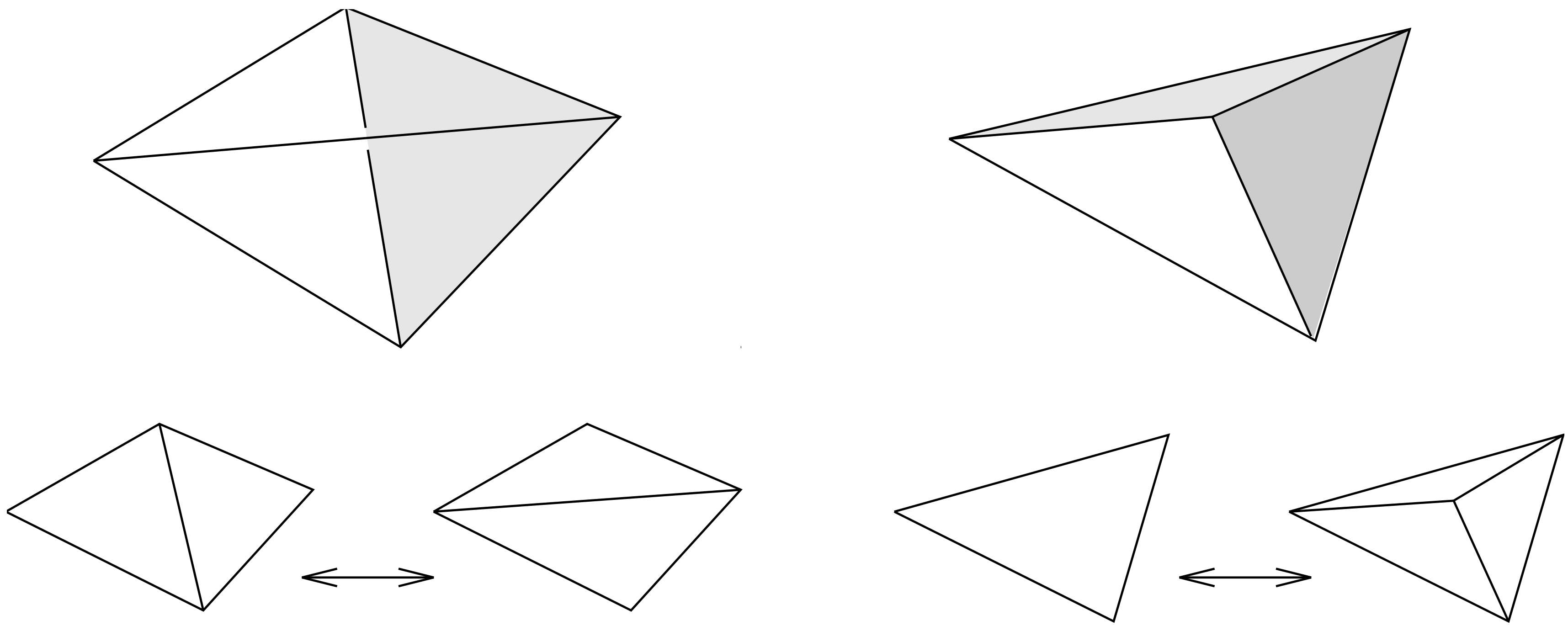
Can't repair intrinsic mesh problems (e.g. high degree vertex)
need operations that change mesh structure

Flipping

Small set of local connectivity-changing operations
Applied in greedy fashion to improve mesh

Simplest case: triangle mesh. Two types of flip:

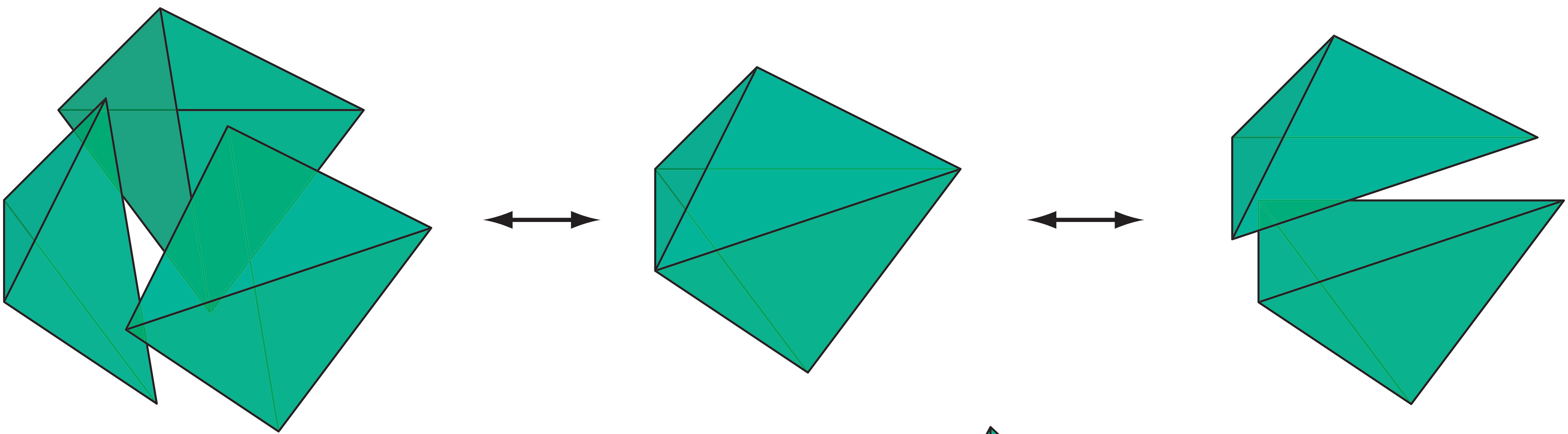
switch diagonal of quadrilateral (2-2)
add/remove degree three vertex (1-3 or 3-1)

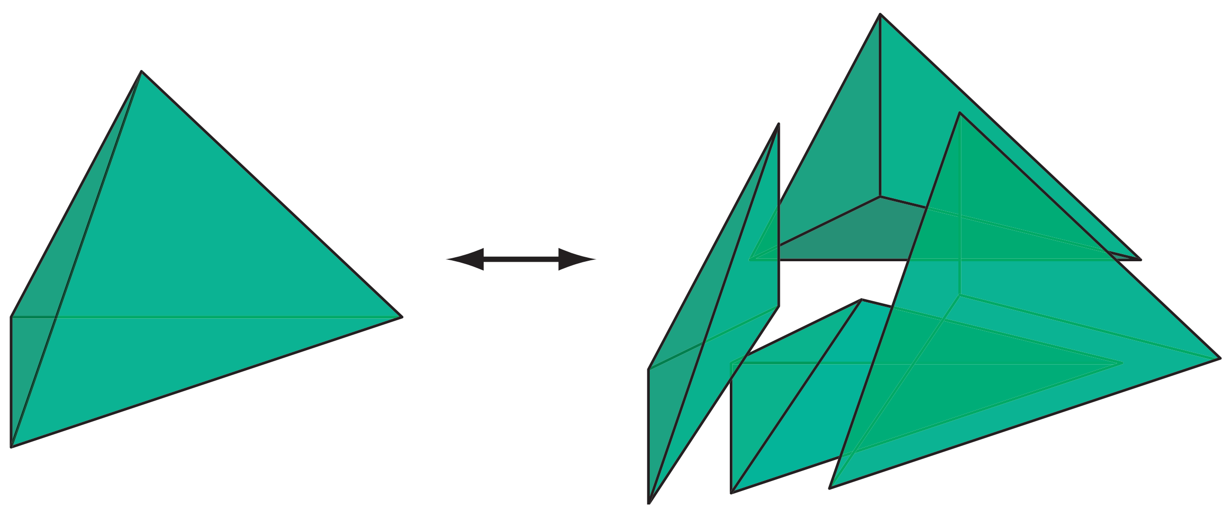


Initial and final configurations of a flip
can be viewed as projections of the bottom and top faces of a tetrahedron

so flipping = gluing tetrahedron onto top of 3d "history mesh"
having the desired 2d mesh as its top surface

Tetrahedron mesh flips: 2-3 or 1-4

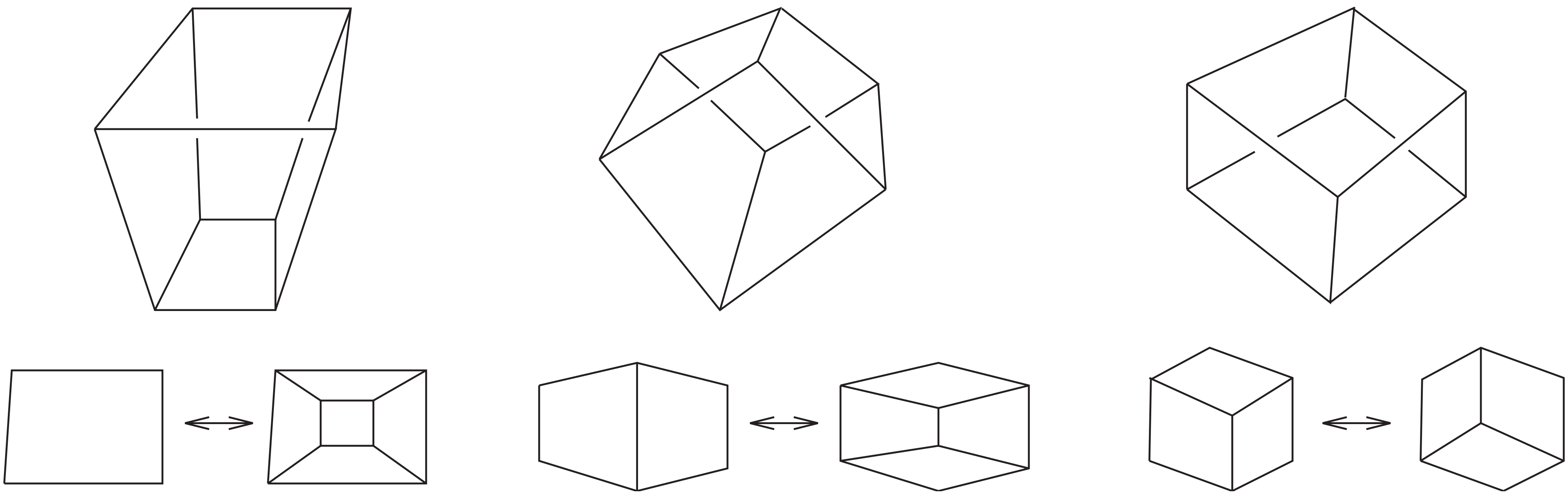




Can be viewed as swapping top/bottom views of 4d simplex
Similar sets of flips generalize to any dimension

Quadrilateral flips

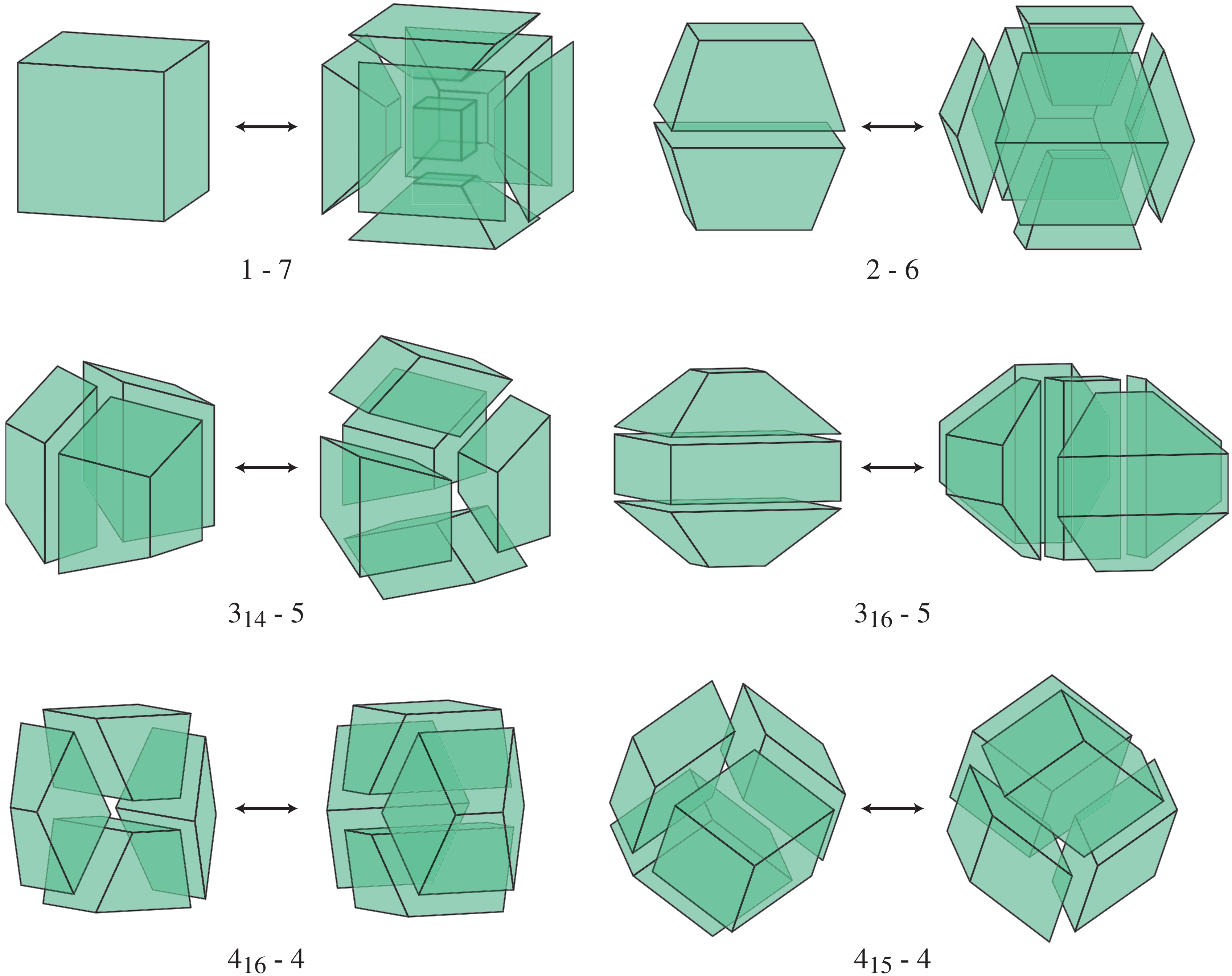
By analogy to triangle/tetrahedron flips,
define as swapping top/bottom views of a cube



Possibilities:

One quad split into five or vice versa
Two quads replaced by four or vice versa
Three quads turned into three rotated quads

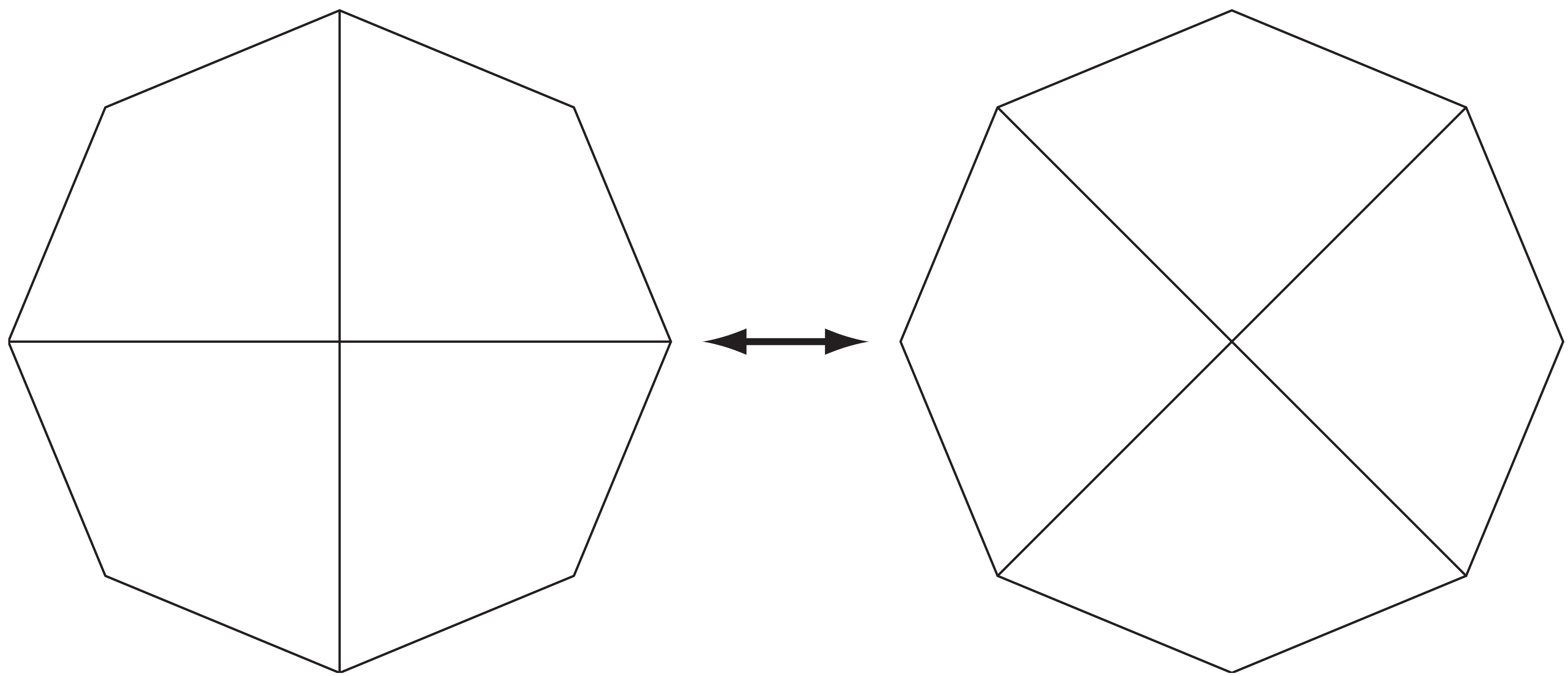
Hexahedral flips



Are flips enough?

I.e. can they substitute for any other local replacement?

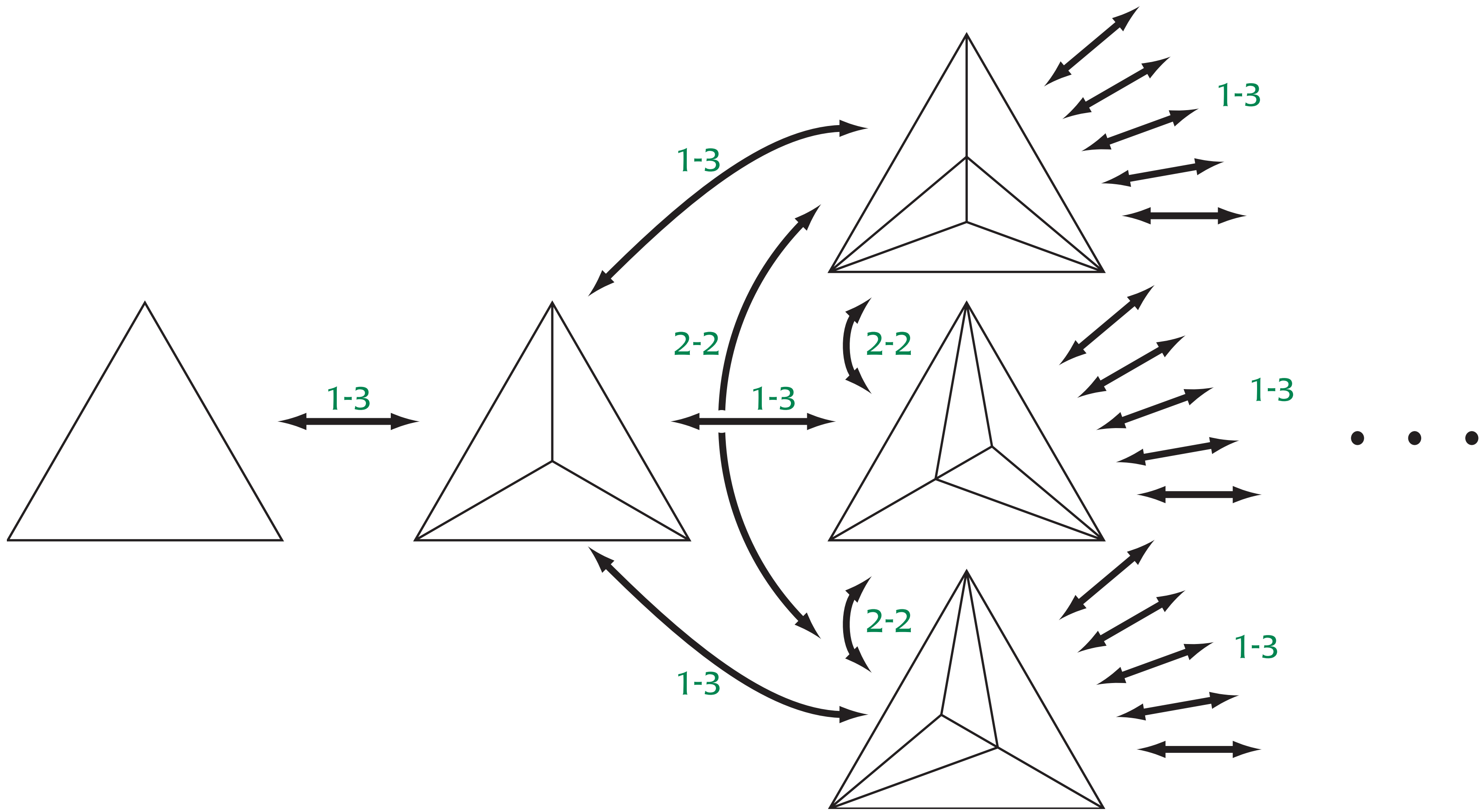
A difficult example:



Flip Graph

vertices = meshes on some domain, edges = flips between meshes

Always connected for triangles, (topological) tetrahedra
open whether connected for geometric tetrahedra



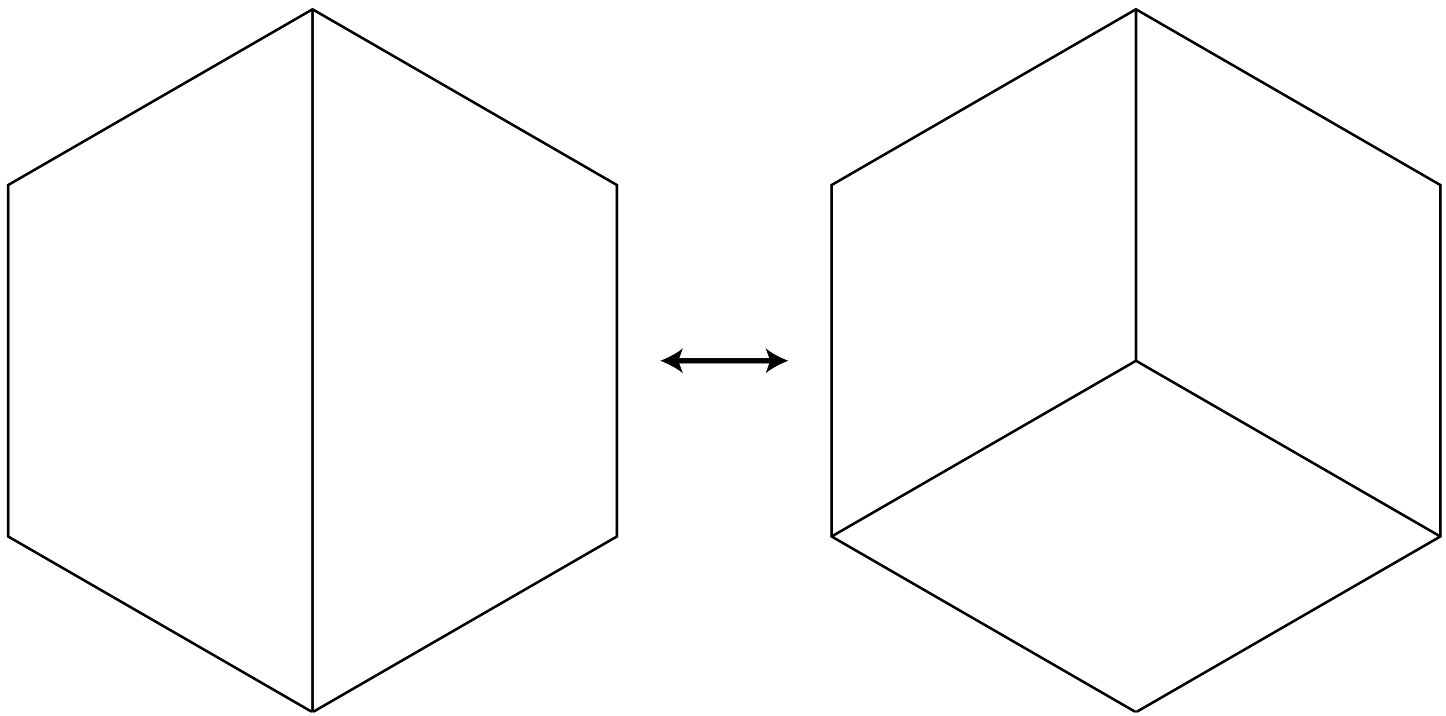
Is flip graph connected for quadrilaterals, hexahedra?

Possibly different answers for different domains, topological vs geometric meshes

Flips preserve parity

Cube and hypercube have even numbers of facets
so quadrilateral and hexahedral flips always replace odd-odd or even-even

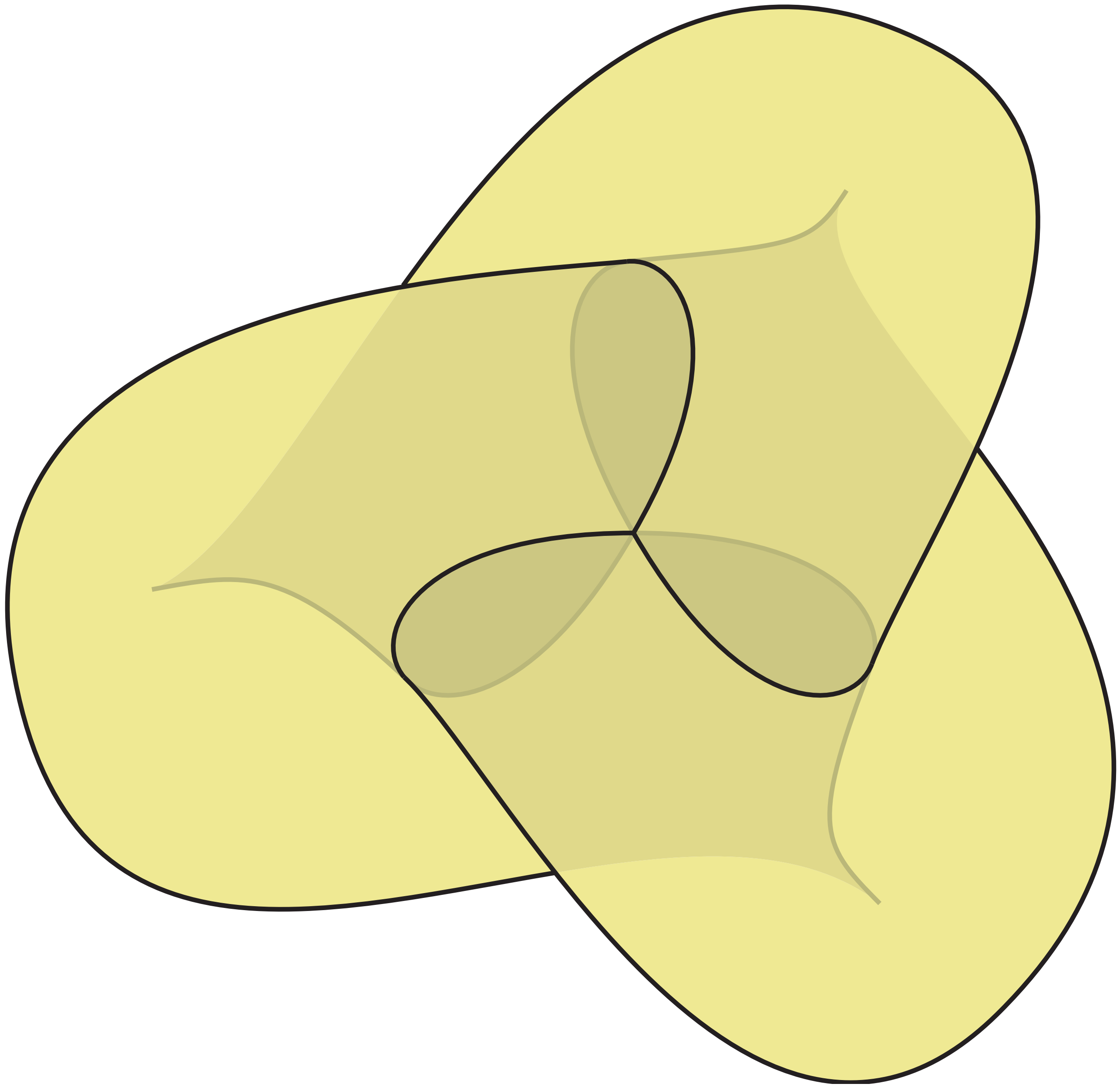
But same domain can have both odd and even meshes:



So flip graph is not connected

How to change parity in hexahedral meshes

Add a copy of Boy's surface to dual surface arrangement



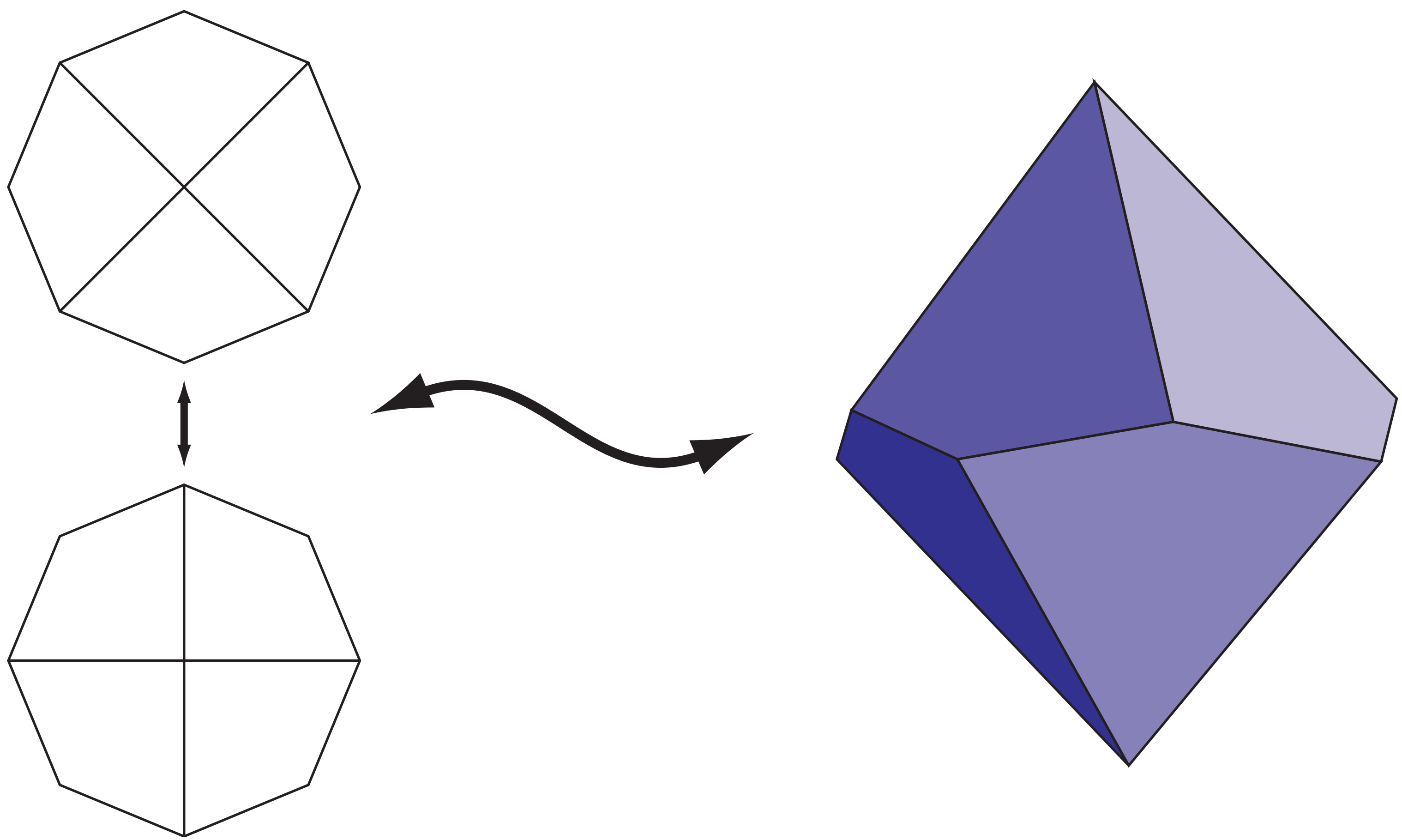
One new hex from self-triple-intersection, even number from intersections w/other surfaces

...but parity is the only obstacle to flipping!

Theorem [Bern and Eppstein, 2001]:

Any equal parity quadrilateral meshes on a simply connected domain can be connected by a sequence of flips

Proof idea: View two meshes as top and bottom surfaces of a 3d domain



Use a hexahedral mesh to determine set of flips
BUT flip sequence ~ shelling, so need shellable mesh

More details of connectivity proof

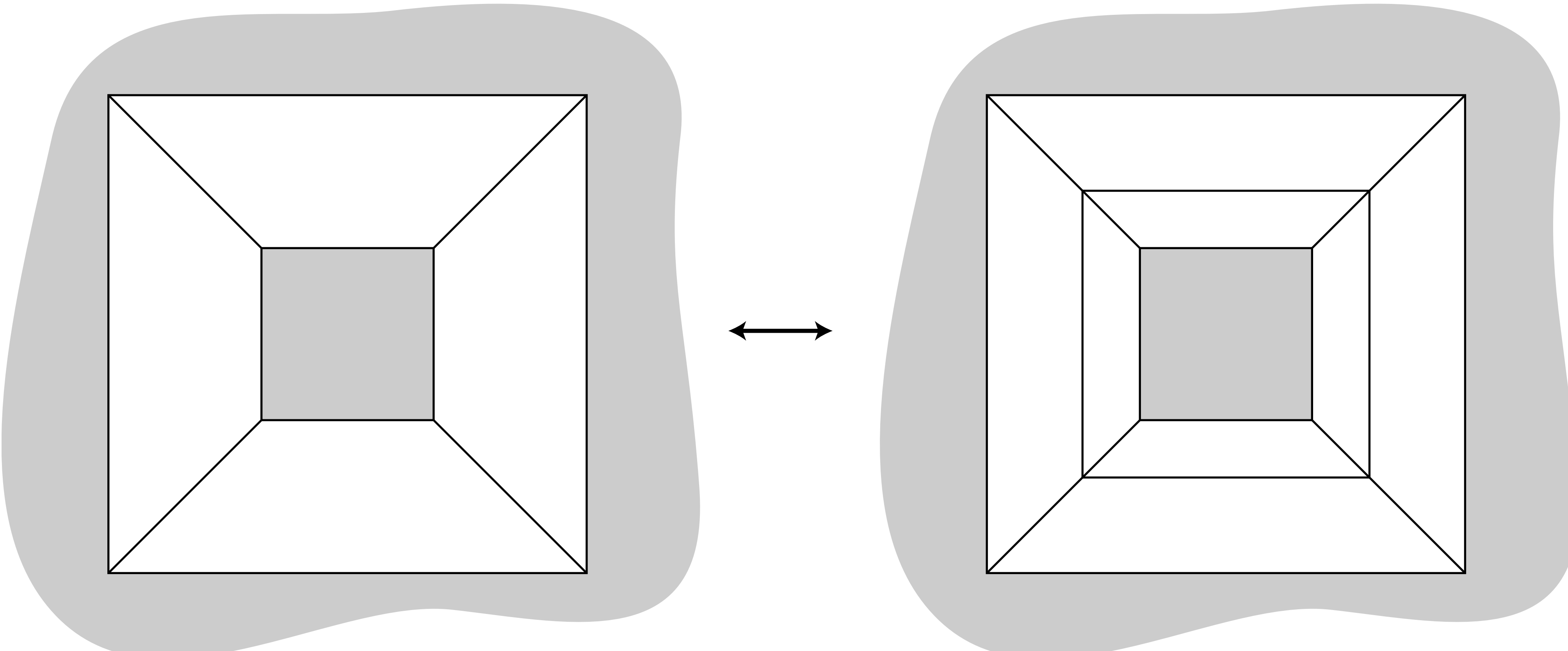
Mesh 3d domain [e.g. via Mitchell & Thurston]
Form dual surface arrangement

Add additional concentric spheres to arrangement
(forming concentric layers of cuboids in mesh)

Drill to center by removing one cuboid per layer
Then remove one layer at a time inside-out
Use drilling + layer removal as shelling/flipping order

Shellability of planar maps allows correct removal of each layer

Simple-connectivity assumption is necessary



Two even-parity meshes of an annular domain

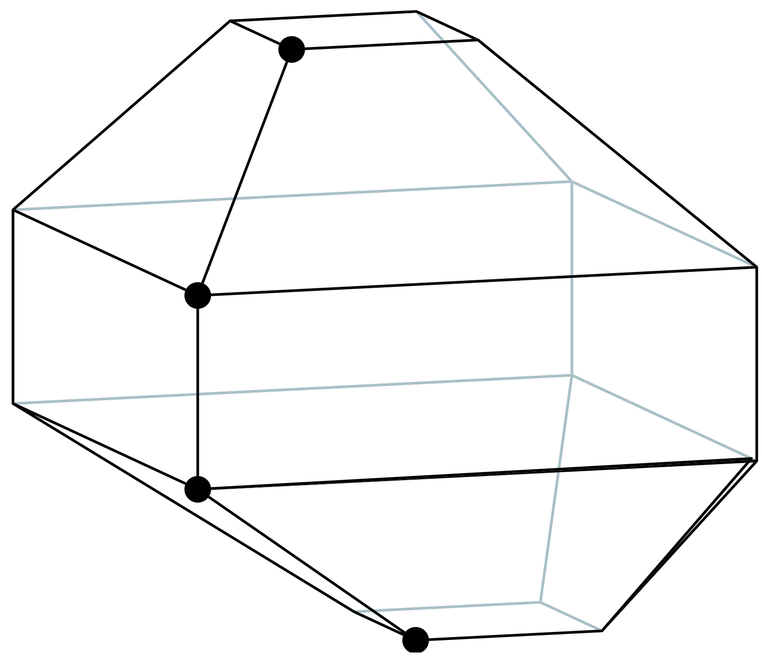
If they could be connected by flips,
flip sequence would give hexahedral mesh of 3 by 4 torus
impossible due to interior triangle

Bicuboid revisited

1-7, 2-6, 6-2, 7-1 flips preserve flatness of facets [Bern and Eppstein, 2001]

Meshes reachable from warped 2-cuboid mesh of bicuboid must also be warped
Rules out many but not all meshes for the bicuboid

However, not all flips preserve flatness:



This polytope has a flat 3-hex mesh but not the flipped 5-hex mesh.

More open questions:

Non-simply-connected 2d domains?

Classify connected components of quad-mesh flip graph

Since all local changes can be simulated by flips,
some non-local changes are needed – what is a good set?

3d flip graph connectivity?

Can use same idea of lifting dimensions and using mesh to guide flips

Need to understand which 4d domains have hypercube meshes

How to use topological methods (dual surfaces etc) to control element shape not just connectivity?