CS 261: Data Structures

Week 6–7: Binary search

Lecture 7a: Augmented search trees

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Ranking and unranking

In sorted arrays

Rank(x) = the position of x in the array
(or the position it would go if added to the array)

Can be found by binary search

Unrank(i) = the element at position i in the array

Trivial to compute as Array[i]

For example, Unrank($n/2$) is the median

They are inverse operations:

- Rank(Unrank(i)) = i, if i is in the range of array indexes
- Unrank(Rank(x)) = x, if x is one of the values stored in the array

In dynamic binary search trees

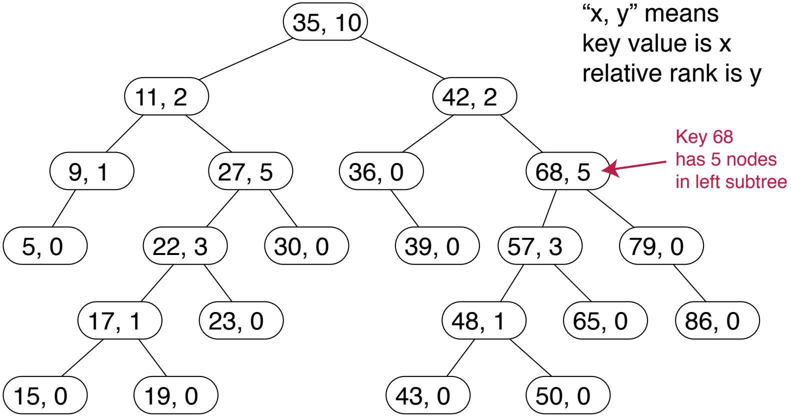
Rank and Unrank are well defined as the position of a given value in the sorted order, and the value at a given position

But it's not obvious how to compute them quickly!
It doesn't work to translate array search directly to trees

- In array binary search for Rank(x), we know the rank of each array cell
- In binary search trees, we cannot store a rank in each tree node, because each update would cause all later ranks to change, too many for fast updating
- There is no way to translate the trivial array Unrank algorithm into a tree algorithm

Augmented binary search trees

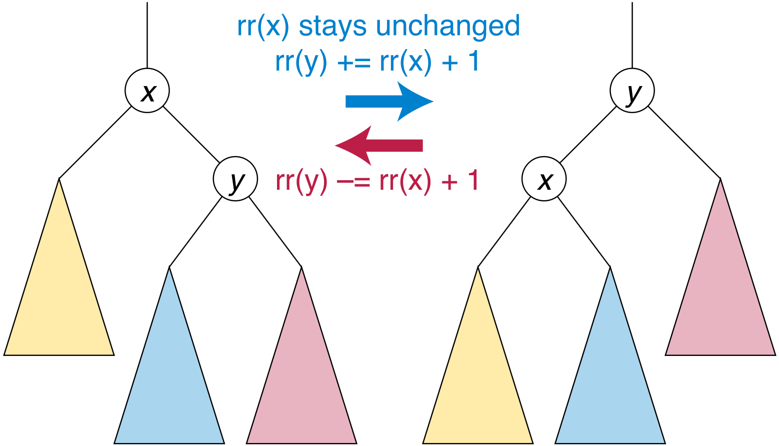
Store relative rank in each node: its position among it and its descendants = number of left descendants



Maintaining relative rank

On insertion or deletion: add or subtract one to all right ancestors

On rotation:



Ranking using relative ranks

Call the following recursive search with node = tree root:

```
def rank(x,node):
    if node == None:
        return 0
    else if x <= node.key:
        return rank(x,node.left)
    else:
        return rank(x,node.right) + node.relrank + 1
```

(In splay trees, add splay from last internal node on search path)

Unranking using relative ranks

Call the following recursive search with node = tree root:

```
def unrank(i,node):
    if i == node.relrank:
        return node.value
    else if i  < node.relrank:
        return unrank(i,node.left)
    else:
        return unrank(i - node.relrank - 1, node.right)
```

(In splay trees, add splay from last internal node on search path)

Ranking and unranking summary

By adding extra information (relative rank) to each node of a binary search tree, we can still update the tree in $O(\log n)$ time, and answer rank and unrank queries in the same time

Works with any rotation-based balanced binary search tree

Related recent research: Ranking and unranking dynamic sorted sets of n integers in the range $[0, n^c]$ can be done slightly faster: $O(\log n / \log\log n)$ per update or query

Pătrașcu and Thorup, "Dynamic Integer Sets with Optimal Rank, Select, and Predecessor Search", FOCS 2014, https://arxiv.org/abs/1408.3045

Range searching

Range searching

Find aggregate information about data elements within a query range [low,high] of values

(or within higher-dimensional regions)

- Range counting: Number of elements in range

  Compute ranks of left and right range endpoints and subtract

- Range reporting: List all elements in range

- Range minimum: Find minimum priority value in range

  (not minimum value – trivial as successor of left endpoint)

- Other more complex queries e.g. do a recursive range search on another attribute for elements within range

Range reporting

Call with node = tree root:

```
def report(low,high,node):
    if low < node.value:
        report(low,high,node.left)
    if low <= node.value <= high:
        output node.value
    if node.value < high:
        report(low,high,node.right)
```

Analysis of range reporting

Whenever we recurse into both children, we also output the node value

Every recursive call is one of:

- A node whose value is output
- A node on the search path for the low range endpoint (at which we search only the right child)
- A node on the search path for the high range endpoint (at which we search only the left child)

Time $= O(\text{number of nodes searched}) = O(\text{output size} + \log n)$

An algorithm whose time depends on output size and not just on input size is called "output sensitive".

Decomposable range search problems

Suppose:

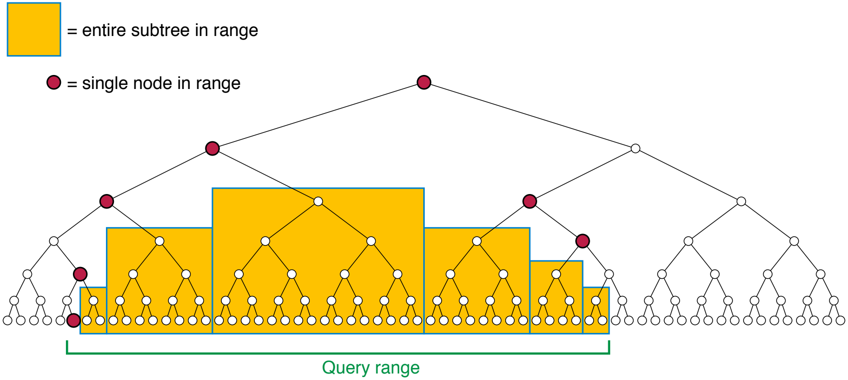
- We have a collection of key,value pairs with sorted keys
- An associative binary operation $\oplus$ operates on the values
- We want to find the result of applying $\oplus$ to the values whose keys are within a query range [low,high]

If we can decompose a range into disjoint sets, $S \cup T$, we can use $\oplus$ to combine results for each set: total $=$ result$(S) \oplus$ result(T)

Examples:

- Range counting, value $= 1$, $\oplus =$ addition
- Range reporting, value$(x) = \{x\}$, $\oplus =$ set union
- Range minimum, value $=$ priority, $\oplus =$ minimization

Partition of range into subtrees



Idea: search paths for range endpoints have length $O(\log n)$

We can decompose the range into $O(\log n)$ nodes on these two paths and $O(\log n)$ entire subtrees between them

Store $\oplus$ for each subtree, combine stored results for query total

Decomposable query algorithm

As we recurse, replace range endpoints by flag values $-\infty$ and $+\infty$ in subtrees for which endpoints are no longer relevant

Whole tree is in range when both endpoints are infinite

To query range [low,high] at a given node:

- If low $= -\infty$ and high $= +\infty$, return stored value for subtree
- If key $>$ high, return query(low, high, left child)
- If key $<$ low, return query(low, high, right child)
- Return query(low, $+\infty$, left child) $\oplus$
  node's value $\oplus$ query($-\infty$, high, right child)

Time: $O(\log n)$ for operations with $\oplus$ time $O(1)$

Maintaining the stored subtree values

Whenever a node's stored subtree value might have changed

- We added or removed a descendant
- It was involved in a rotation

Recompute its subtree value as

left subtree value $\oplus$ right subtree value $\oplus$ node's value

Time per insertion or deletion $O(\log n)$

(under same assumptions on $\oplus$ time as for query)

Works for any balanced binary search tree

Range query summary

Using augmented search trees, we can:

Answer range counting or range minimization in time $O(\log n)$

Answer range reporting in time $O(\log n + \text{output})$

Handle insertions or deletions in time $O(\log n)$

Generalize to other decomposable range searching problems

Lower bounds

Lower bounds on data structures

We have seen:

- Optimality of binary heap for comparison-model priority queues

  Based on the ability to sort using heaps

  Can be sidestepped by using integer arithmetic and array indexing instead of only comparisons (e.g. flat trees)

- Impossibility of nontrivial set disjointness

  Based on unproven assumption (SETH)

This time: Lower bounds for range search

Proven rigorously in a very general computational model

Are augmented search trees optimal?

We have seen that a very general class of dynamic range searching problems can be solved in time $O(\log n)$

Natural question: Is that the right time bound or can we do better?

Answer: we can prove $\Omega(\log n)$, for:

- Simple and natural range searching problem: range sum

  Data = ordered keys and numeric values

  Query = sum of values for key-value pairs with key in range

- A very general model of computing: cell probe model

  Only measure communication between CPU and memory

Warmup interview question: Static range sums

You are given an array of n numbers

Problem: process it so you can quickly find the range sum

$$A[i] + A[i+1] + \cdots + A[j-1] + A[j]$$

Solution

Store an array of prefix sums

$$PS[i] = \sum_{j=0}^{i} A[j] = A[0] + A[1] + \cdots A[j] = PS[i-1] + A[i]$$

Return $PS[j] - PS[i-1]$

Linear space and preprocessing, constant time per query

Prefix sum problem

Simplified version of the range sum problem

(for lower bounds, simpler problem $\Rightarrow$ stronger bound)

Maintain array $A[0] \ldots A[n-1]$ of numbers

Update(i, x): set $A[i]$ to new value x

Query(i): calculate prefix sum $A[0] + A[1] + \cdots + A[i]$

(If these operations are hard, so are the more general operations of insertion + deletion + range sum)

Log-time solution

Build a perfectly balanced binary tree with array A at its leaves

Each internal node stores sums of its two children

Query(i): sum up left children on search path to $A[i]$

Update: recompute node sums on path to root

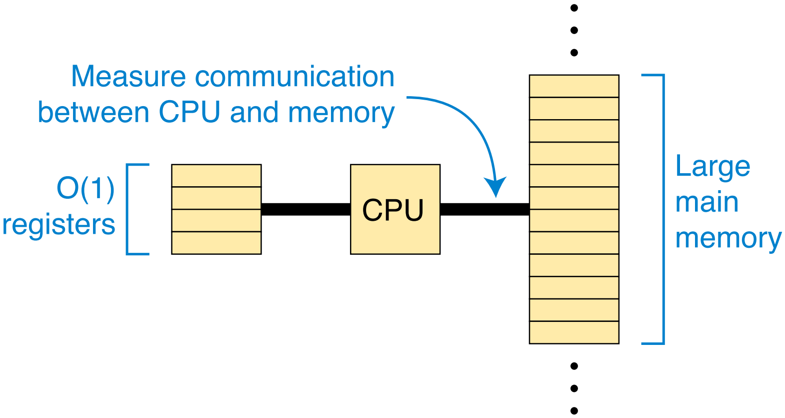
Claim: No other data structure can achieve better O-notation

We need to define what an "other data structure" might be

Cell probe model of computing

Central processor has $O(1)$ registers, each holding one word (binary value of length $w \geq \log_2 n$); memory has up to 2^w words

We count only steps that move a word between CPU and memory $\Rightarrow$ lower bound doesn't depend on what other steps are allowed



Fitting prefix sums to cell probe model

We are going to prove a lower bound for
prefix sums of n w-bit binary numbers

(representation size of the input values should be
the same as the word size of the computer)

We will use $n =$ a power of two (unrelated to word size)

To avoid questions of integer overflow,
we will assume all arithmetic is modulo 2^w

(just do binary addition and ignore overflows)

Goal: Find a sequence of prefix sum operations that forces any correct data structure to
do a lot of CPU–memory communication

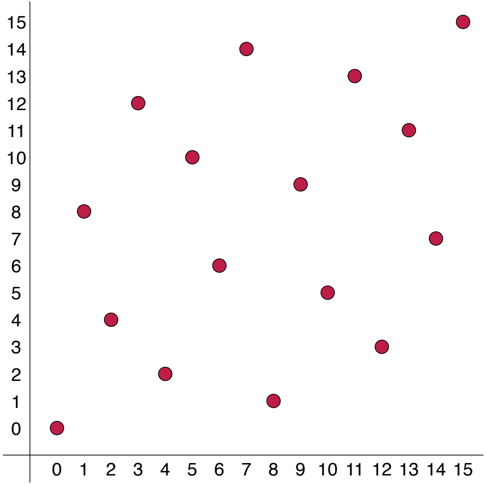
A special permutation of n

Assume $n = 2^k$

Define "bit reversal permutation" $r(i)$:

- Write i as a k-bit binary number
- Reverse the bits
- Interpret the result as a binary number

E.g. for $k = 8$,
$222_{10} = 11011110_2$
becomes
$01111011_2 = 123_{10}$



Computing sequence of bit-reversals

To compute a sequence of length 2^k, consisting of all k-bit numbers in bit-reversed order, compute the same sequence recursively for $k-1$ and use it twice:

```
def bitrev(k):
    if k == 0:
        return [0]
    L = bitrev(k-1)
    return [2*x for x in L] + [2*x+1 for x in L]
```

$[0] \to [0,1] \to [0,2,1,3] \to [0,4,2,6,1,5,3,7] \to \ldots$

Each value in the second half of the sequence is one plus the corresponding value in the first half

A difficult sequence of prefix-sum operations

Initialize all data values $A[i]$ to zero, then:

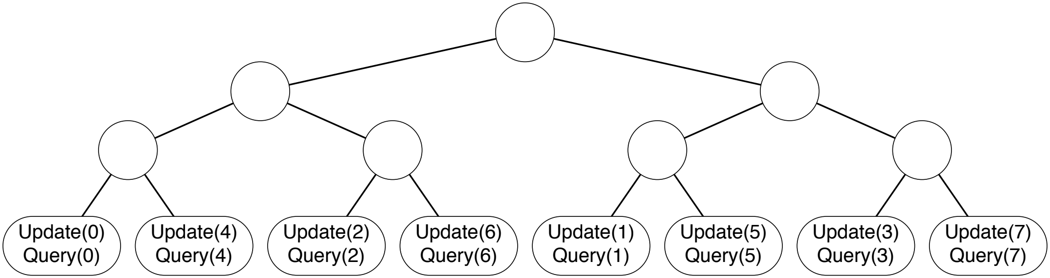
For each index i in `bitrev[k]`:

- Set $A[i]$ to be a random w-bit number
- Query the prefix sum $A[0] + \cdots + A[i]$

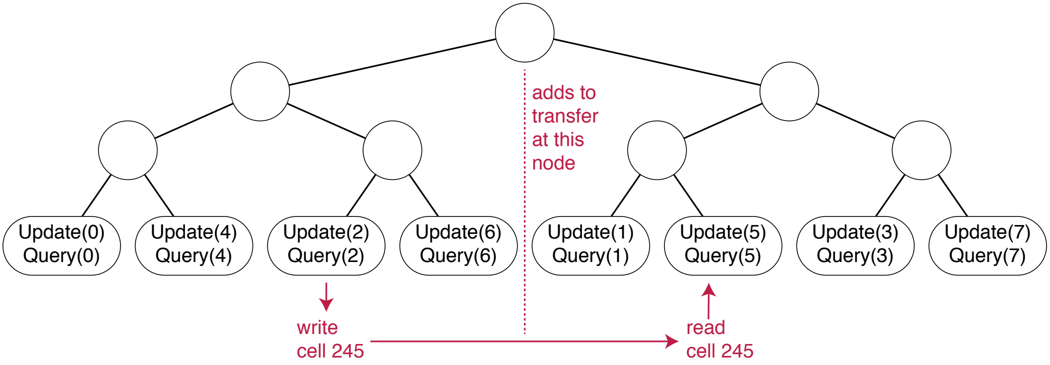
E.g. when $n = 8$, $k = 3$, we perform the operations
Update(0,random), Query(0), Update(4,random), Query(4),
Update(2, random), Query(2), Update(6,random), Query(6),
Update(1,random), Query(1), Update(5,random), Query(5),
Update(3,random), Query(3), Update(7,random), Query(7)

A binary tree on the sequence of operations

This is not a data structure! It's just a mathematical tree that we will use in the lower bound proof.



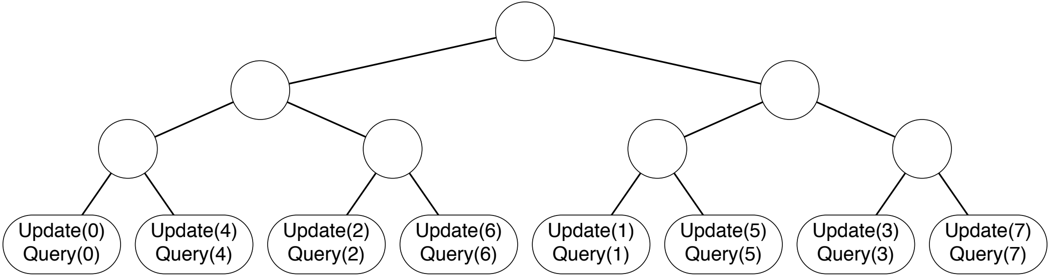
Information transfer



For any data structure for prefix sums, and any node x of this tree, define the information transfer of x to be the number of times an operation in the right descendants of x reads a memory cell that was last written during the operations in the left descendants of x

Each memory read contributes to information transfer at ≤ 1 node $\Rightarrow$ total number of read steps $\geq$ total information transfer

Information transfer $\geq$ descendants/2



Information transfer = number of times an operation in node's right descendants reads a memory cell last written on the left

Let d = #descendants/2 = # left updates = # right queries

There are 2^{wd} different possible values for the updates on the left, each of which would produce different query results on the right

(Independently from information derived from non-transfer reads)

$\Rightarrow$ for correct queries, information transfer $\geq d$

Finishing the lower bound

Information transfer at root node of tree: $\geq n/2$

Information transfer at ith level of tree:
2^i nodes with transfer $\geq n/2^{i+1}$, total $\geq n/2$

Total over whole tree: $\geq (n/2) \times \#$ levels $= (n/2)\log_2 n$

There are $2n$ prefix sum operations (updates and queries together) $\Rightarrow$ average number of memory reads per operation $\geq \frac{1}{4}\log_2 n$

Every prefix sum data structure that fits into the cell probe model of computation requires $\Omega(\log n)$ time per operation

$\Rightarrow$ same is true for dynamic range sum data structures