CS 261: Data Structures

Week 4: Streaming and sketching

Lecture 4a: Medians and sampling

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Streaming and sketching

The main idea

Sometimes data is too big to fit into memory...

Sketching

Represent data by a structure of sublinear size

Preferably $O(1)$; at most $n^{1-\varepsilon}$ for some $\varepsilon > 0$

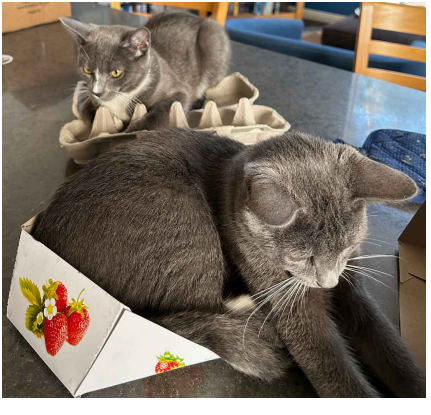
Keep enough information to solve what you want either exactly or approximately

Streaming

Read through data once, in some given order

Use a sketch to represent information about it

Produce a result from the sketch



Example: Sketch for the average

Sketch for the average of a collection of numbers: pair (n, t)

n is the number of elements in the collection
t is the sum of the elements

Operations:

- Incorporate new element x into collection:
  Add one to n and add x to t
- Remove element from collection:
  Subtract one from n and subtract x from t
- Compute average of collection:
  Return t/n

Total space is $O(1)$ regardless of how big n is

(Time/operation is also $O(1)$ but our main concern is space)

Example: Streaming average

To compute the average of a sequence S:

- Initialize an empty sketch for the average
- For each x in the sequence, incorporate x into the sketch
- Return the average of the the sketch

Time for an n-element sequence is $O(n)$
Working space (not counting the space for the input) is $O(1)$

This works even when the sequence is generated somehow in a way that does not involve local storage (e.g. internet traffic)

Cash register versus turnstile

Two different variations of streaming algorithms:

Cash register:

Money goes in, doesn't come back out

Meaning for streaming:

Input has insertions, no removals

Turnstile:

People can either enter or exit

Meaning for streaming:

Input has both insertions and removals

Streaming average was presented in cash register model
but the same sketch also works for turnstile model

Beyond streaming

Sketches can sometimes be combined to produce information about multiple data sets

Example: Can produce an average-sketch for the concatenation of two sequences by adding the sketches of the two sequences

If we have n sequences, of total length N, we can compute all pairwise averages in time $O(n^2 + N)$, faster than computing each pairwise average separately

Medians and reservoir sampling

Medians

Median: Sort given list of n items; which one goes to position $n/2$?

Like average, an important measure of the central tendency of a sample, more robust against arbitrarily-corrupted outliers

Applications in facility location: Given n points on a line, place a center minimizing the average distance to the given points

Impossibility of streaming median

No streaming algorithm can compute the median!

Consider sketch after seeing the first $n/2$ items in a stream

Depending on the remaining items, any one of these first $n/2$ could become the median of the whole stream

So we have to remember all their values

Space is $\Omega(n)$, not $O(n^{1-\varepsilon})$ for some $\varepsilon > 0$

Approximate medians

We want the item at position $\frac{1}{2}n$ in sorted sequence

Relaxation: δ-approximate median
Position should be between $(\frac{1}{2} - \delta)n$ and $(\frac{1}{2} + \delta)n$

Easy randomized method:

- Maintain a sample of $\Theta\big((1/\delta)^2\big)$ sequence members
- Return the median of the sample

Chernoff $\Rightarrow$ constant probability of being in desired position
Bigger constant in sample size $\Rightarrow$ better probability

(Later this lecture: a non-random method that is always within the desired approximation bound, but more complicated and with non-constant space complexity)

More detailed analysis of approximate medians

Let sample size be $2c/\delta^2$

Random event: one of our samples is less than the median

Expected number of random events: c/δ^2.

Our estimate is too small when actual number of events is larger by a factor of $1 + 2\delta$

Chernoff (case for multiplier close to one): probability this happens is $\leq e^{-(2\delta)^2 \cdot c/\delta^2 \cdot \frac{1}{3}}$

The deltas cancel $\Rightarrow$ constant probability of success; symmetric for estimate too small

Increase c for greater constant; use $c \approx \log n$ for probability $\approx 1 - \frac{1}{n}$

Reservoir sampling

How to maintain a sample of k random elements (without replacement) from a longer sequence?

Store an array A of size k, and the number n (initially 0)
For each element x:
    If $n < k$:
        Store x in $A[n]$
    Else:
        Choose a random integer i from 0 to n
        If $i < k$, store x in $A[i]$
    Increment n

More careful: one random number per sample (when to replace) instead of per input

MinHash

Hashing for reservoir sampling

We saw how to maintain a sample of k elements, using a random choice for each element

Instead, make a single random choice, of a hash function $h(x)$

Sample = the k elements with the smallest values of h

Two advantages:

- Less use of (possibly slow) random number generation
- Repeatable: if different agents use the same hash function to sample the same sets, they will get the same samples

We can take advantage of repeatability to use this for more applications!

[Broder 1997]

Combining sets with MinHash

To compute the union:

$$\text{MinHash}(S \cup T) = \text{MinHash}\big(\text{MinHash}(S) \cup \text{MinHash}(T)\big)$$

Because any element that is one of the k smallest hashes in the union must also be one of the k smallest in either of the smaller sets that contain it

We can get useful information about sets S and T from this computation!

Key question: how many elements from $\text{MinHash}(S) \cup \text{MinHash}(T)$ survive into $\text{MinHash}(S \cup T)$?

Original motivation for MinHash

Before the success of Google, a different company (DEC) ran a different popular search engine (AltaVista)

They had a problem: Many web pages can be found in multiple similar (but not always identical) copies

Search engines should return only one copy of the same page, so user can see other results without getting swamped by duplicates

$\Rightarrow$ Need to quickly test similarity of web pages

From documents to sets

"Bag of words" model: represent any web page by the set of words used in it

Similar documents (web pages) should have similar words

So after converting web pages to bags of words, the search engine problem becomes: quickly test similarity of pairs of sets

Jaccard similarity

A standard measure of how similar two sets are:

$$J(S, T) = \frac{|S \cap T|}{|S \cup T|}$$

Scale-invariant (normalized for how big the two sets are)

Ranges from 0 to 1

- 0 when sets are disjoint
- close to 0 when sets are not very similar
- close to 1 when sets are similar
- 1 when sets are identical

Estimating Jaccard similarity

Theorem:

$$J(S,T) = \frac{1}{k} E\Big[\big|\text{MinHash}(S \cup T) \cap \text{MinHash}(S) \cap \text{MinHash}(T)\big|\Big]$$

The intersection consists of the elements of $S \cap T$ that also belong to $\text{MinHash}(S \cup T)$

Linearity of expectation $\Rightarrow$

$$\begin{aligned} E[\dots] &= \sum_{x \in S \cup T} \Pr[x \in \text{intersection of MinHashes}] \\ &= \sum_{x \in S \cup T} \Pr[x \in S \cap T] \times \Pr[x \in \text{MinHash}(S \cup T)] \\ &= \sum_{x \in S \cup T} \frac{|S \cap T|}{|S \cup T|} \times \frac{k}{|S \cup T|} \\ &= |S \cup T| \times J(S,T) \times \frac{k}{|S \cup T|} = k\, J(S,T) \end{aligned}$$

How accurate is it?

Suppose we use

$$\frac{1}{k}\left|\text{MinHash}(S \cup T) \cap \text{MinHash}(S) \cap \text{MinHash}(T)\right|$$

as an estimate for $J(S, T)$

Previous analysis shows that this has the correct expected value:
it is an unbiased estimator

Our analysis decomposed the estimate into a sum of independent 0-1 random variables (is each member of $S \cup T$ in the intersection)

$\Rightarrow$ Can use Chernoff bounds for being within $1 \pm \varepsilon$ of expectation

To get expected error $|J(S, T) - \text{estimate}| \leq \varepsilon$, set $k = \Theta(1/\varepsilon^2)$

Summary of MinHash

Main idea: sketch any set by choosing the k elements that have the smallest hash values

Can reduce randomness in reservoir sampling

Sketches of unions can be computed from unions of sketches

Provides accurate estimates of Jaccard similarity

All-pairs similarity of n sets with total length N: time $O(n^2 + N)$

Deterministic sampling

Small sketches exist (but can we find them?)

Given a sequence S of n numbers (allowing repetitions), define the quantile $q_S(x)$ of a number x to be the fraction of sequence elements that are less than x

(So median = element whose quantile is $1/2$)

ε-approximation: subsequence $\hat{S}$ of S such that $q_{\hat{S}} \approx q_S$:

$$\forall x \quad |q_{\hat{S}}(x) - q_S(x)| \leq \varepsilon$$

It's easy to construct an ε-approximation of size $1/\varepsilon$: just choose $1/\varepsilon$ numbers equally spaced in the sorted sequence

The median of an ε-approximation would accurately approximate the median of S

Composition of epsilon-approximations

Concatenation

If we concatenate two equal-length sequences S and T to form $S\,T$, and both have equal-length ε-approximations $\hat{S}$ and $\hat{T}$, then $\hat{S}\,\hat{T}$ is a ε-approximation of $S\,T$

But now it's twice as long

Approximation of approximation

A δ-approximation of the ε-approximation $\hat{S}\,\hat{T}$ is a $(\delta + \varepsilon)$-approximation of size $1/\delta$

Worse approximation but short again

Idea: Use concatenation to combine approximations from subsequences and then use approximation to shorten the result

Hierarchical sketching, basic idea

For each block B of 2^i consecutive numbers in the input stream, store an approximation, computed as follows:

- Break B into two blocks of 2^{i-1} consecutive numbers
- Concatenate the approximations stored for these two blocks
- Compute an approximation of the concatenation

  (Merge two sorted lists and then take evenly-spaced items from the result)

Hierarchical sketching, parameterized

Choose an approximation parameter ε_i for each i

For each block B of 2^i consecutive numbers in the input stream, store an approximation, computed as follows:

- Break B into two blocks of 2^{i-1} consecutive numbers
- Concatenate the ε_{i-1}-approximations stored for these two blocks
- Compute a $(\varepsilon_i - \varepsilon_{i-1})$-approximation of the concatenation

  (Merge two sorted lists and then take evenly-spaced items from the result)

Updating a hierarchical sketch

To insert a new element:

Create a new block of one element ($i = 0$)
(even if there is already another one-element block)

While two blocks have the same size 2^i:

- Merge them into one block of size 2^{i+1}
- Concatenate their ϵ_i-approximations
- Construct a $(\varepsilon_{i+1} - \varepsilon_i)$-approximation of the concatenation

Once we have merged two blocks, forget their approximations
(We only store the remaining unmerged blocks, at most one of each size)

Using a hierarchical sketch

To approximate the quantile of any value x:

- Use the approximations to find $q_B(x)$ for each block
- return the weighted average (weighted by block size)

To find an ε-approximate median:

- Find x whose approximate quantile is near $1/2$. (Can be found using a weighted median in the union of block approximations.)
- Its position in the whole input is within $\varepsilon\, n$ positions of the median

How big are these sketches?

For any $\delta > 0$, the sum

$$\sum_{i=0}^{\infty} \frac{1}{(i+1)^{1+\delta}}$$

has a finite sum s_δ

Set

$$\varepsilon_{i+1} - \varepsilon_i = \frac{\varepsilon}{s_\delta} \cdot \frac{1}{(i+1)^{1+\delta}}$$

$\Rightarrow$ sequence of parameters ε_i converges in telescoping sum to desired ε

Summing resulting sketch sizes for all blocks gives space $O\left(\varepsilon^{-1}(\log n)^{2+\delta}\right)$

[Bagchi et al. 2007]

References and image credits

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Sara Eppstein. Smoke in a strawberry box and Mist in an egg carton.

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