CS 164 & CS 266: Computational Geometry

Lecture 4

Triangulation

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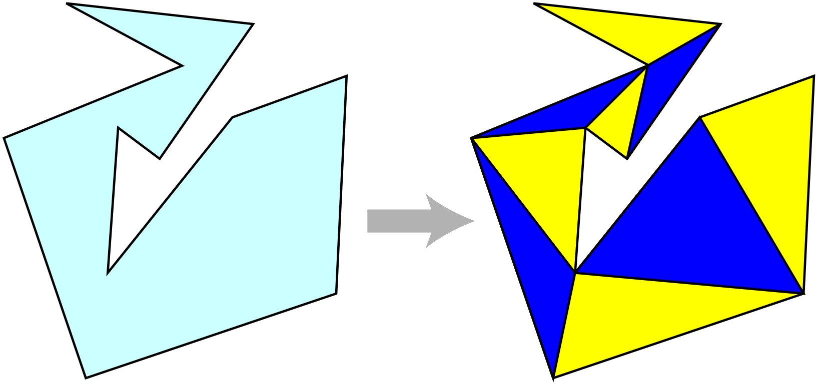
Existence of triangulations

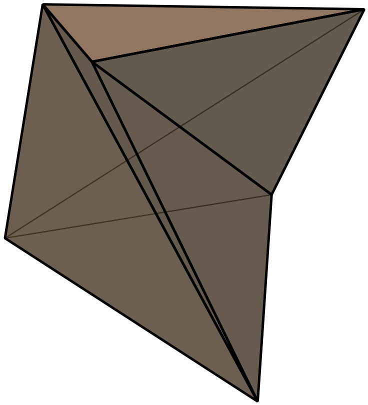
What is a triangulation?

Subdivide polygon (possibly with polygonal holes) into edge-to-edge triangles

Not allowed to add new vertices



3d polyhedra do not always have triangulations



This shape is called the Schönhardt polyhedron [Schönhardt 1928]

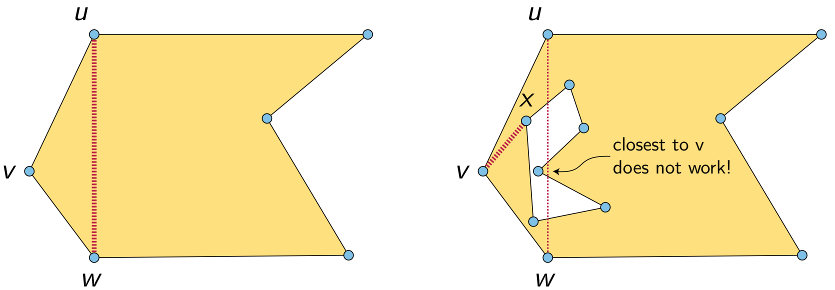
For every four vertices, the tetrahedron they form extends outside the polyhedron

So it cannot be subdivided into face-to-face tetrahedra

2d polygons always have triangulations

True more generally for non-crossing arrangements of segments

Proof idea: If not already triangulated, can add one more segment



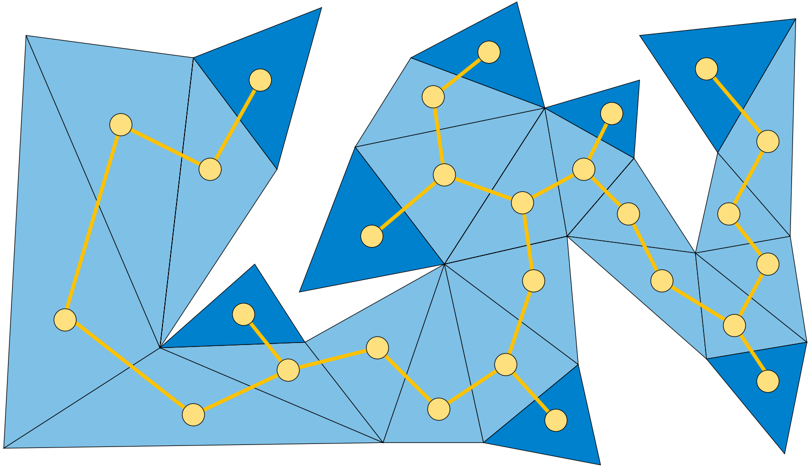
Let v be leftmost vertex of a non-triangle face, neighbors u–v–w

If we can add edge uw, do it

Else something must be inside triangle uvw blocking visibility; add edge vx where x is inside triangle and farthest from line uw

Trees and ears

For a simple polygon (meaning no holes), the triangles of any triangulation are adjacent in a tree pattern, because any cycle would surround an interior vertex or a hole



Every tree has a leaf ⇒ every triangulation of a simple polygon has an ear, a triangle that uses two polygon edges

The art gallery problem

The art gallery problem

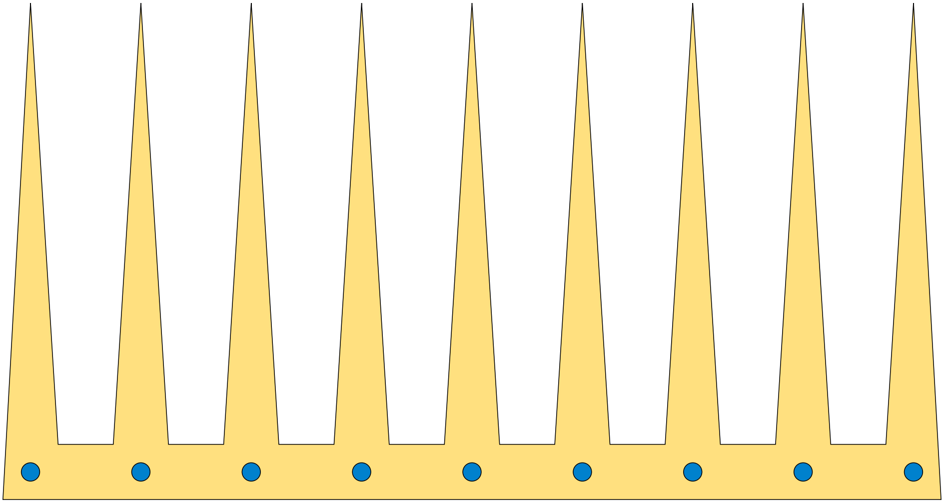
Given an art museum with a complicated floor plan

Position enough guards (or cameras) to see everything

[Daderot 2019]

Hard-to-guard galleries

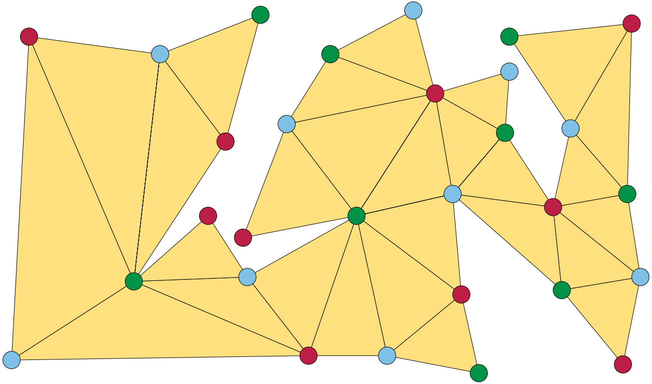
Some simple polygon floor plans with n sides require $\left\lfloor \frac{n}{3} \right\rfloor$ guards



27 sides, 9 guards

Coloring triangulations

Vertices of any triangulation of a simple polygon can be colored by three colors so that every triangle has one vertex of each color



Proof: Remove an ear, color the rest recursively / by induction, put the ear back and color tip differently than its two neighbors

Takes linear time once you have the triangulation

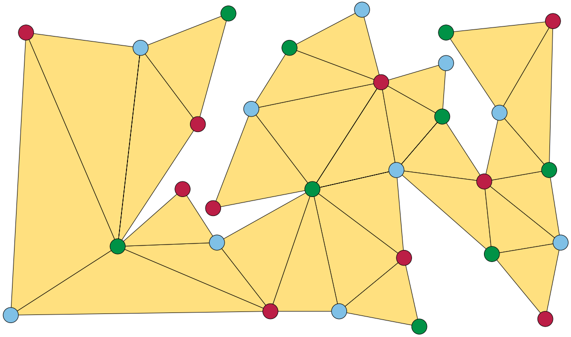
The art gallery theorem

Every simple polygon can be guarded by $\leq \left\lfloor \frac{n}{3} \right\rfloor$ guards

[Chvátal 1975; Fisk 1978]

Proof: Guard vertices of the least-frequent color

Every triangle is guarded by one of its three vertices



10 red, 9 green, 10 blue $\Rightarrow$ guard with 9 green vertices

Fewer guards may be possible! NP-hard to find them [Abrahamsen et al. 2022]

Triangulation algorithms

How to triangulate a polygon?

Simple polygons can be triangulated in linear time [Chazelle 1991]
but the algorithm is completed and impractical

The book gives:

An $O(n \log n)$ plane sweep algorithm for partitioning into monotone polygons
(polygons for which every vertical line crosses the boundary ≤ 2 times)

A linear-time algorithm for triangulating monotone polygons, still complicated

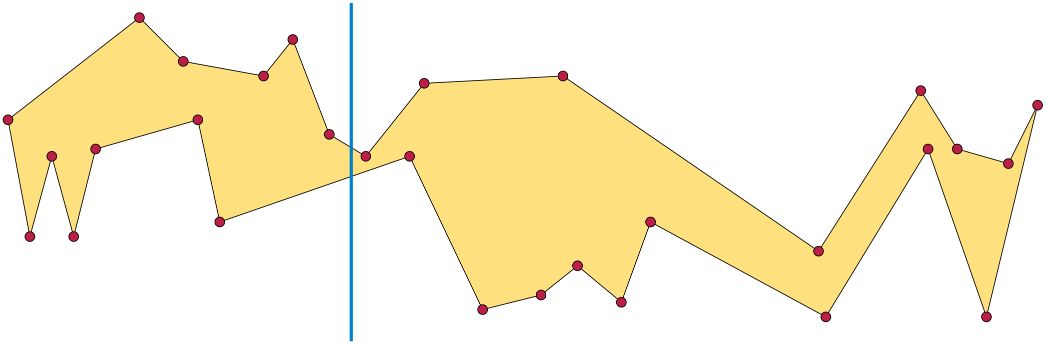
This idea of finding special classes of polygons with linear-time algorithms
was made obsolete by Chazelle's result

Still open: Find a practical linear-time triangulation algorithm

Simpler $O(n \log n)$ triangulation

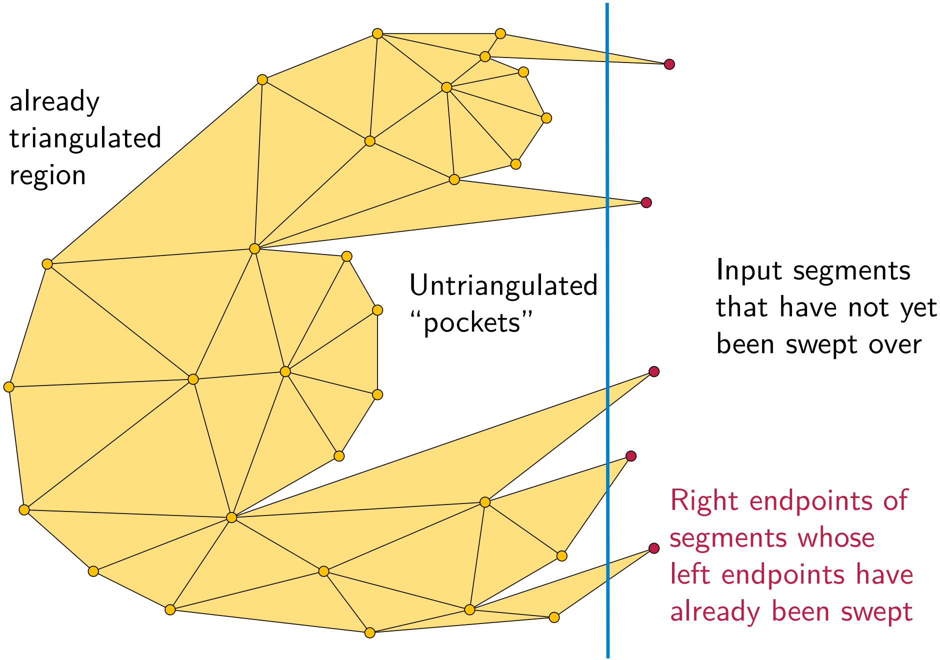
A more direct $O(n \log n)$ plane sweep is possible

It works for any non-crossing set of segments (not just polygons)

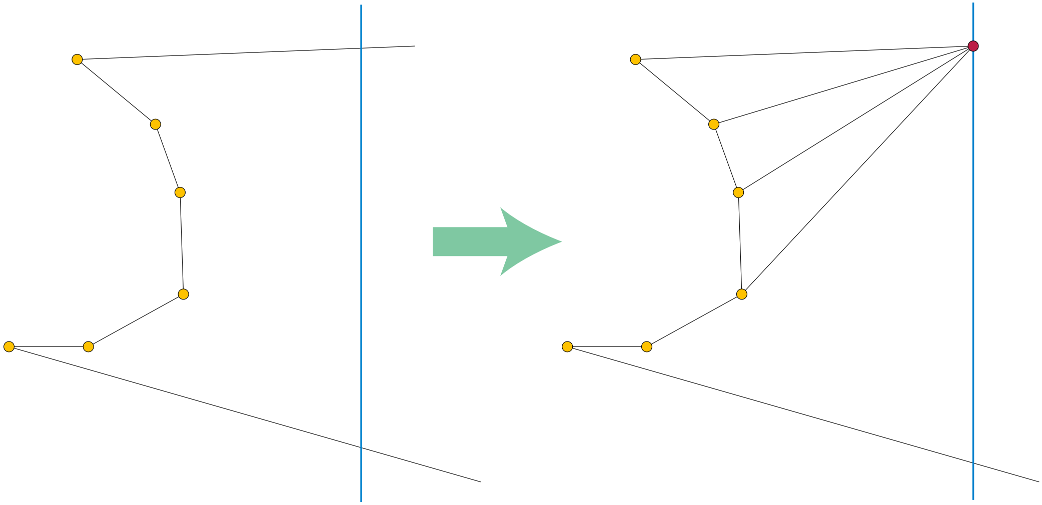


For each vertex in left-to-right order, connect it to everything farther to the left that it can still see after earlier steps

State in the middle of a sweep



The main idea

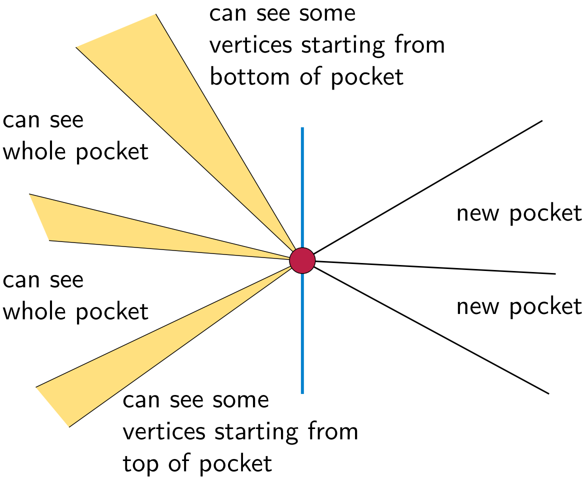


When we sweep over a vertex that can see into a pocket, triangulate what it can see

$\Rightarrow$ pocket becomes smaller

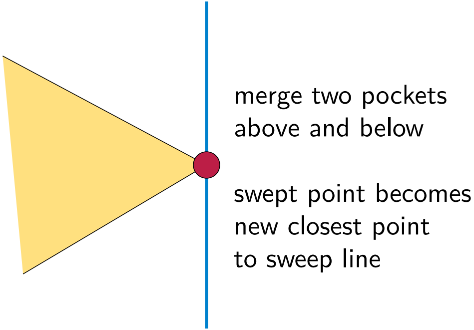
Easy cases

Vertex that is both a left endpoint of one or more input segments and a right endpoint of one or more input segments

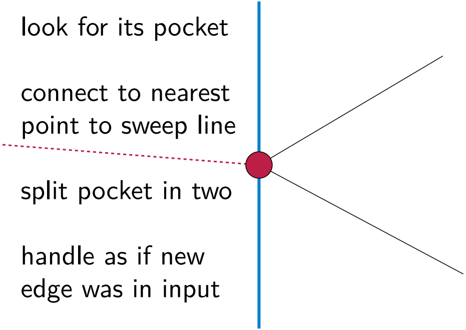


Messier cases

Swept vertex is a right endpoint of some segments from the input, but not a left endpoint of any



Swept vertex is a left endpoint of some segments from the input, but not a right endpoint of any



How to represent this state

- Binary search tree of pockets, ordered by the vertical ordering in which they cross the sweep line
- Doubly-linked list of the already-swept vertices in each pocket
- Pointer into this list for each pocket, to the vertex closest to the sweep line

Pseudocode

- Initialize binary search tree of pockets
- Sort the input segment endpoints by x
- For each endpoint, in sorted order, and for each wedge formed by two segments touching it, do one of the following cases:
  - Wedge angle $< \pi$ to the right (easy)
  - Wedge angle $< \pi$ to the left (easy)
  - Left endpoint of one segment, right endpoint of other (easy)
  - Wedge angle $> \pi$, right endpoint of both segments (messy)
  - Wedge angle $> \pi$, left endpoint of both segments (messy)

References and image credits

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