

CS 163 & CS 265: Graph Algorithms

Week 4: More paths

Lecture 4b: Euler tours

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Overview

Walks that visit everything in the graph

Tour: start and end at same vertex

Undirected graphs

“Everything”: May mean all vertices or all edges

But most of this also works for directed graphs and for walks that start and end at different vertices

Four variants of the problem

Euler tour (today)

Unweighted, walk that uses each edge exactly once

Linear time

Chinese postman tour

Weighted, shortest walk that uses each edge at least once

Named after Meigu Guan

Polynomial time

Hamiltonian cycle

Unweighted, cycle that uses each vertex exactly once

NP-complete

Traveling salesperson tour

(next week)

Weighted, shortest walk that visits each vertex at least once

NP-complete

The Seven Bridges of Königsberg

Königsberg

- ▶ City on the Baltic Sea coast, at the mouth of the Pregel river
- ▶ Name means “king’s mountain”
- ▶ Formerly in Prussia (pre-WWII Germany)
- ▶ Taken over by Russia in 1945, renamed Kaliningrad
- ▶ Disconnected from the main part of Russia
- ▶ Home of the Baltic Fleet of the Russian Navy

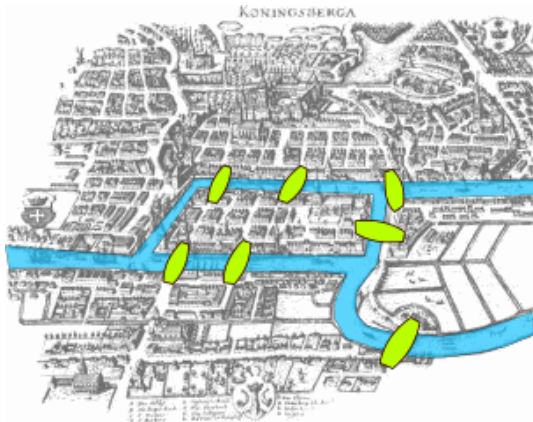


CC-BY-SA image Kaliningrad map (1)
from Wikimedia commons

The Seven Bridges of Königsberg

Early-1700s Königsberg had seven bridges, connecting two islands with the river banks

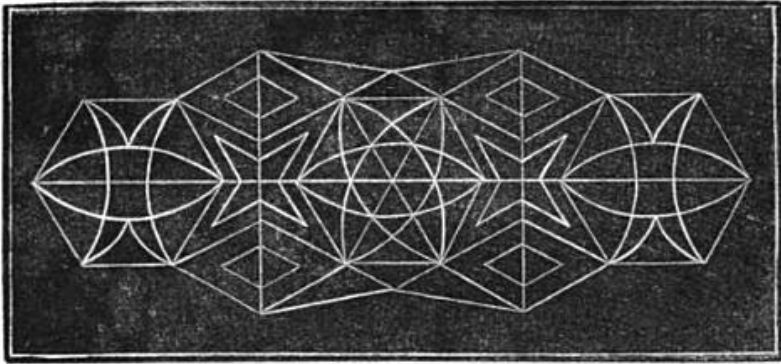
A puzzle from the time asked whether one could walk through the city crossing each bridge exactly once



Mathematician Leonhard Euler published a solution in 1736 that became the first published work in graph theory [Euler 1736]

Another more complicated example

Puzzle: Can you draw this pattern using a pen or pencil, as a single continuous line, without lifting your pen or pencil from the paper?

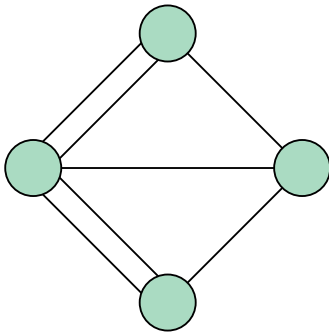
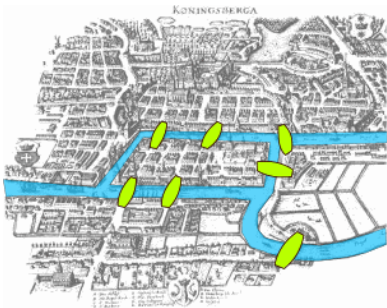


[Listing 1848]

Translation into a graph problem

Make a vertex for each piece of land and an edge for each bridge

(It's actually a multigraph because some pieces of land are connected by two bridges)



Problem becomes: Does some walk use each edge exactly once?

Degree and the handshaking lemma

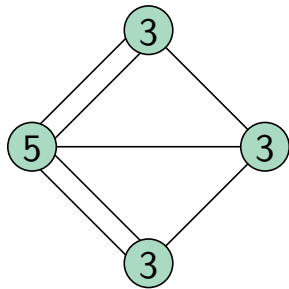
Degree of a vertex: how many edges touch it?

Odd vertex: a vertex whose degree is an odd number

Euler's handshaking lemma:

Every finite graph has an even number of odd vertices

(The Königsberg graph has four)



Proof idea: Each edge has 2 endpoints $\Rightarrow$ the whole graph has $2m$ vertex-edge contacts, an even number

Counting by vertices instead of edges, $\# \text{contacts} = \sum_v \deg(v)$.

Every even vertex contributes an even number of contacts, and every odd vertex contributes an odd number, so for the sum to be even the number of odd vertices must be even

Necessary conditions for an Euler tour

Whenever a tour enters and then leaves a vertex, the two edges it crosses contribute an even number to the degree of the vertex

When a tour starts at a vertex or ends at a vertex, the edge at the start or end of the tour contributes an odd number to the degree

So if an Euler tour (through each edge exactly once) starts and ends at the same vertex, all vertices must have even degree

If the tour starts and ends at different vertices, those two vertices have odd degree and all the rest must be even

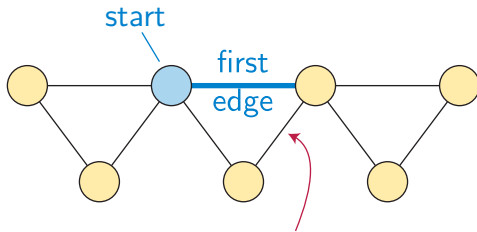
Königsberg has four vertices of odd degree, too many $\Rightarrow$ no tour

Euler claimed that every connected finite graph with zero or two odd vertices has a tour, but did not provide a proof

Algorithms

Fleury's algorithm [Fleury 1883]

- ▶ A good method for pencil and paper but slow & complicated for computers
- ▶ Start anywhere when all degrees are even, or at an odd vertex when there are two odd vertices
- ▶ Greedy algorithm: keep adding edges to the end of the curve, but don't make mistakes
- ▶ Mistake: taking the last edge connecting two uncovered parts of the graph



choosing this next would be a mistake!

Why does Fleury's algorithm work correctly?

When a path does not end at the goal, its endpoint has an odd # of uncovered edges

⇒ any path can be extended to reach the goal (at least one more edge exists)

⇒ after you make a mistake, you could still reach the goal

⇒ mistake edge is a “bridge” separating part of remaining graph from goal

Just before you make a mistake, you are at a vertex with more than one uncovered edge
(the mistake edge + the other edges that get separated from the goal)

But there can be only one bridge otherwise graph would not become separated

⇒ every other edge we could take from that vertex is not a mistake

⇒ until all edges covered, a non-mistaken next edge always exists

How fast is Fleury's algorithm?

Naïve implementation

Use reachability to test whether each choice is a mistake

If it is, choose any other edge (no need to test again)

$O(m)$ each time we add an edge to the tour, $O(m^2)$ total

Complicated data structures

Use a data structure that can keep track of the uncovered edges and tell which ones disconnect the rest, in small time per test

Randomized $O(m \log n (\log \log n)^3)$ [Thorup 2000]

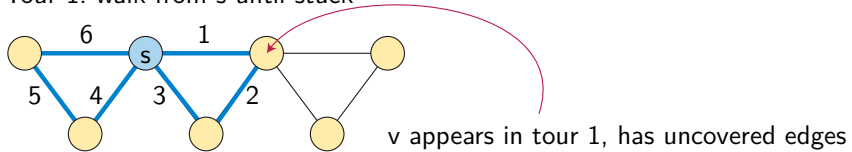
Non-random $O(m(\log n)^2 / \log \log n)$ [Wulff-Nilsen 2013]

Hierholzer's algorithm [Hierholzer 1873]

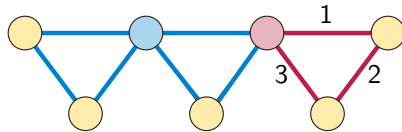
- ▶ Choose a starting vertex and then form tour one edge at a time, like Fleury
- ▶ Keep following edges until getting stuck, but without checking for making mistakes
- ▶ Then, while there exists vertex v on the current tour at which some edges are still uncovered:
 - ▶ Start a new tour that follows uncovered edges in the same way from v until getting stuck at v again
 - ▶ Splice the new tour into the position of v of the main tour

Hierholzer example

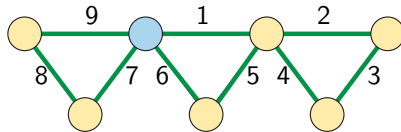
Tour 1: walk from s until stuck



Tour 2: walk from v until stuck



Insert tour 2 at the position of v in tour 1



Linear-time implementation details

- ▶ Decorate each edge with a Boolean, is it used yet?
- ▶ Decorate each vertex with an iterator of its edges
- ▶ Represent the tour in a way that allows efficient insertions into the middle (e.g. a linked list, not a Python list); initialize it to just contain the starting vertex
- ▶ Keep a pointer to the first vertex v on the main tour that might still have uncovered edges, initially s
- ▶ Repeat:
 - ▶ Use the iterator for v to find its next edge vw
 - ▶ If the iterator runs out of edges, advance v to the next vertex on the main tour, or exit if there is no next vertex
 - ▶ Else if vw is already covered, continue
 - ▶ Else start a new tour with edge vw , using the iterators for each vertex to add more edges to the end of the tour until reaching a vertex whose iterators have run out, and then attach the new tour after v in the main tour

A parallel algorithm

More complicated, but allows many things to be done simultaneously:

- ▶ At each vertex, group edges into pairs
- ▶ Following pairs (if you come into a vertex by one edge, go out on its pair) covers the graph by many cycles, but we want only one big cycle
- ▶ Use reachability to find the cycles
- ▶ Make an auxiliary graph whose vertices describe cycles and whose edges describe certain ways of gluing pairs of cycles together
- ▶ Find a spanning tree in the auxiliary graph
- ▶ Use the spanning tree to guide which pairs of cycles to glue

[Awerbuch et al. 1984]

Morals of the story

Sometimes recreational puzzles lead to important mathematics

There are multiple ways of asking for a tour that visits everything, with different complexities

The simplest one, a walk that visits each edge once, has an easy test for whether it exists, and can be found in linear time

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