CS 163 & CS 265: Graph Algorithms

Week 10: Planar graphs

Lecture 10a: Properties

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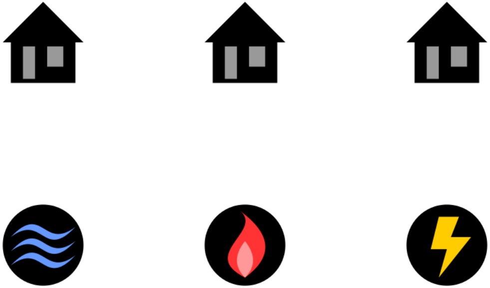
Winter Quarter, 2025

Examples of planar and nonplanar graphs

The three utilities problem

Input: three houses and three utilities, placed in the plane

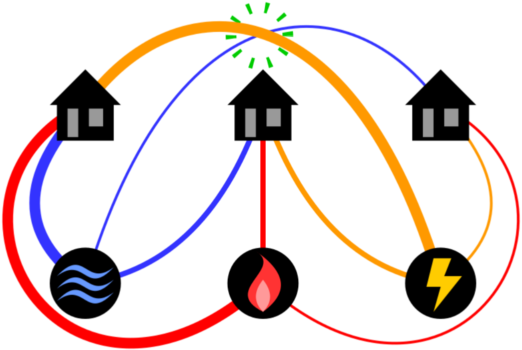


https://commons.wikimedia.org/wiki/File:3_utilities_problem_blank.svg

How to draw lines connecting each house to each utility, without crossings (and without passing through other houses or utilities)?

There is no solution!

If you try it, you will get stuck. . .



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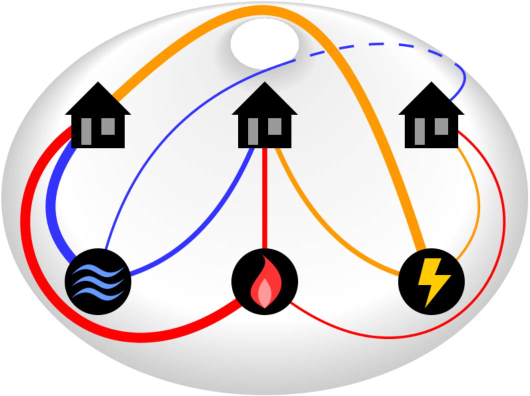
But this is not a satisfactory proof of impossibility

Other surfaces have solutions

If we place the houses and utilities on a torus (for instance, a coffee cup including the handle) we will be able to solve it



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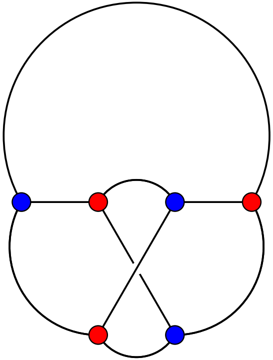
So solution depends somehow on the topology of the plane

Planar graphs

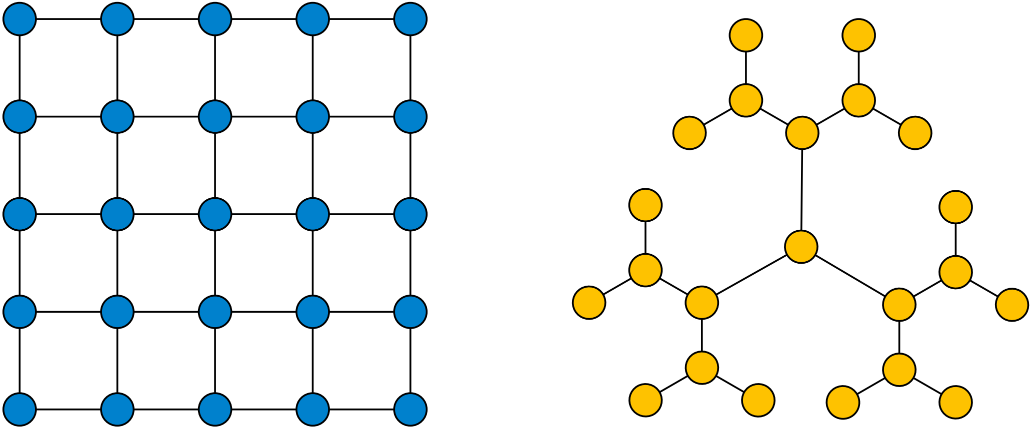
A planar graph is a graph that can be drawn without crossings in the plane

The graph we are trying to draw in this case is $K_{3,3}$, a complete bipartite graph with three houses and three utilities

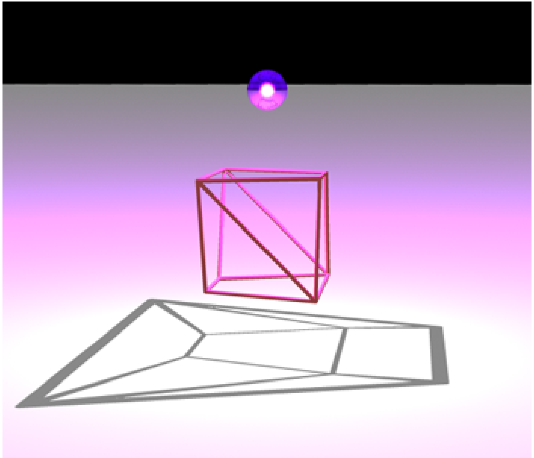
It is not a planar graph; the best we can do is to have only one crossing



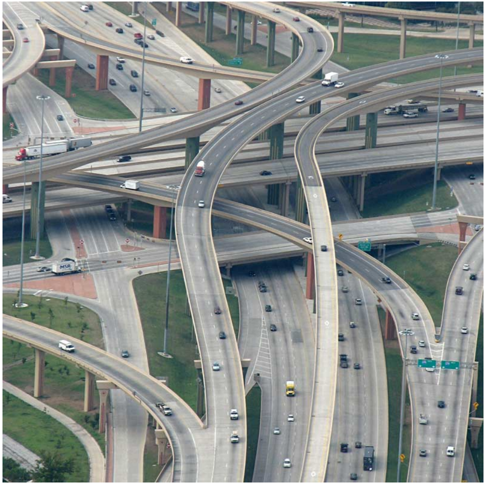
Examples of planar graphs: trees and grid graphs



Examples of planar graphs: convex polyhedra



Mostly but not entirely planar: road networks



https://commons.wikimedia.org/wiki/File:High_Five.jpg

[Eppstein and Gupta 2017]

Properties

Some properties of planar graphs

- **Euler's formula**: If a connected planar graph has $n = V$ vertices, $m = E$ edges, and divides the plane into F faces (regions, including the region outside the drawing) then $V - E + F = 2$
- Simple planar graphs have $\leq 3n - 6$ edges
  $\Rightarrow$ their degeneracy is ≤ 5 (at least one vertex of degree ≤ 5)
- **Four-color theorem**: Can be colored with ≤ 4 colors
  (Appel and Haken [1976]; degeneracy-based greedy gives ≤ 6)
- **Planar separator theorem**: treewidth is $O(\sqrt{n})$
  (Lipton and Tarjan [1979] $\Rightarrow$ shortest paths $O(n)$ [Henzinger et al. 1997], shortest unweighted cycle $O(n)$ [Chang and Lu 2013], etc.)
- **Product structure theorem**: Every planar graph is a subgraph of a product $P \boxtimes T$ where P is a path and T has treewidth $O(1)$
  (Dujmović et al. [2020]: "strong product" has a vertex per pair of vertices in P and T and an edge whenever both components of pairs are equal or adjacent)

Some of the history of Euler's formula

Stated in 1537 for the five regular polyhedra by Francesco Maurolico (unpublished) [Friedman 2018]

Related formulas for polyhedra (but not Euler's formula itself) given by René Descartes in 1630, not published, accidentally dumped into the river Seine after Descartes' death, rescued, copied by Leibniz before being lost again, copy also lost, finally rediscovered in 1860 [Federico 1982]

Announced 1750, published 1758 by Euler but with a buggy proof

First correct proof published by Legendre in 1794

There are many proofs; the one we will see (later this lecture) is by von Staudt, 1847

Euler's formula implies few edges

Euler's formula: $V - E + F = 2$
$\iff F = E - V + 2$

Every face has ≥ 3 sides, and every edge forms exactly two sides of faces, so

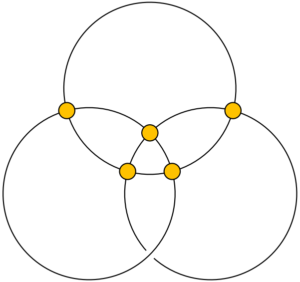
$2E = \#$ sides of faces $\geq 3F$

Use inequality to eliminate F:

$(2/3)E \geq E - V + 2$

Simplify: $E \leq 3V - 6$

$\Rightarrow K_5$ is not planar



$V = 5$ and $E = 10 = 3V - 5$, too many edges

Even fewer for bipartite planar graphs

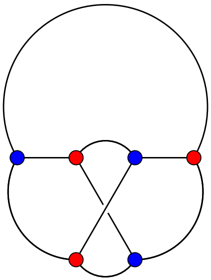
Euler's formula: $V - E + F = 2$
$\iff F = E - V + 2$

In bipartite graphs, every face has ≥ 4 sides, so
$2E = \#$ sides of faces $\geq 4F$

Use inequality to eliminate F:
$(2/4)E \geq E - V + 2$

Simplify: $E \leq 2V - 4$

$\Rightarrow K_{3,3}$ is not planar

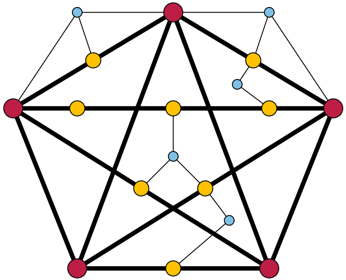


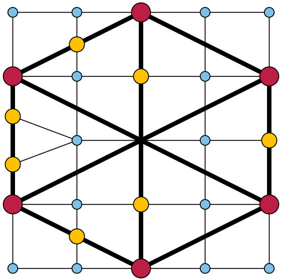
$V = 6$ and $E = 9 = 2V - 3$, too many edges

Kuratowski's theorem

A graph is planar if and only if it does not have a subgraph formed from $K_{3,3}$ or K_5 by subdividing edges into disjoint paths

So a certifying algorithm (week 3) can certify that a graph is planar by drawing it, or that it is not by finding one of these subgraphs





[Kuratowski 1930; Frink and Smith 1930; Menger 1930]

Duality

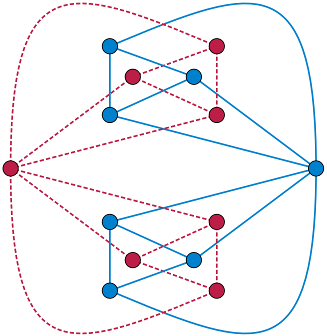
Dual graphs

Start with a “primal” graph, drawn without crossings in the plane

Draw a new “dual” graph with a dual vertex in the middle of each primal face

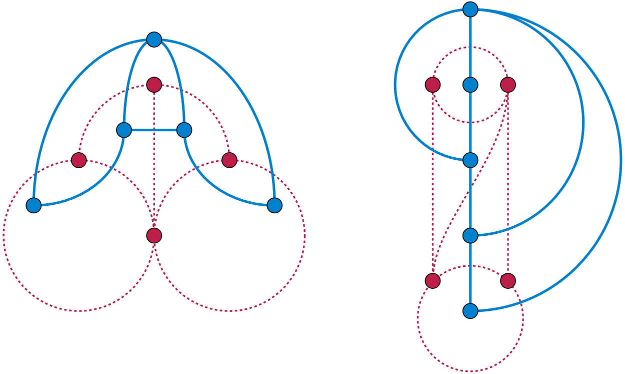
Connect two dual vertices by an edge when the corresponding primal faces share an edge

It’s planar again, and the dual of the dual is the starting primal graph



It’s a property of the drawing, not the graph

A given planar graph may have multiple different ways it can be drawn without crossings, with different duals



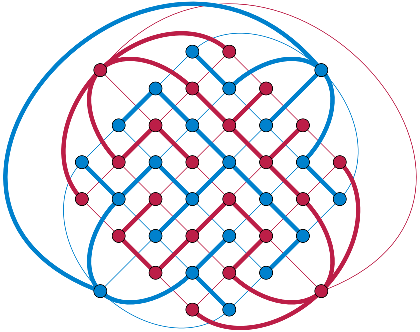
The blue primal graphs are the same; its red duals are different (the left one has a degree-5 vertex; the right one does not)

Duality $\Rightarrow$ Euler

Spanning tree = subgraph with no cycles that connects all the vertices

When a subgraph S has a cycle, it surrounds some faces, disconnecting inside from outside $\Longleftrightarrow$ the dual complement (subgraph of dual formed by dual edges not crossed by S) is a disconnected graph

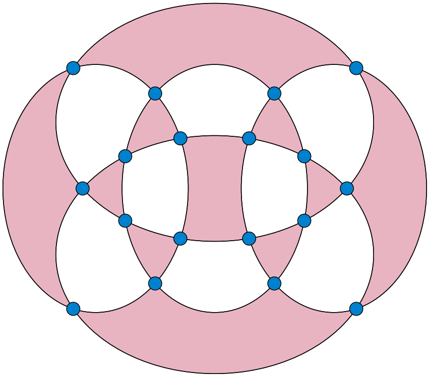
Spanning tree $\Longleftrightarrow$ no cycles and connected $\Longleftrightarrow$ dual complement is connected and has no cycles $\Longleftrightarrow$ also a spanning tree!



Spanning tree has $V - 1$ edges and dual complement spanning tree has $F - 1$ edges, so total is $E = (V - 1) + (F - 1)$

More dual properties

In a planar graph with an Euler tour (all degrees even), color the faces dark if they are surrounded an odd # of times by the tour, light otherwise $\iff$ 2-coloring of the dual graph



Primal graph is Eulerian $\iff$ dual graph is bipartite

Even more dual properties

Can generalize duality to directed planar graphs

Acyclic $\iff$ dual strongly connected

Shortest path $\iff$ minimum cut in dual

(Cuts and flows can be computed faster in planar graphs than in other kinds of graphs)

Morals of the story

Planar graphs are important in applications where we want to visualize graphs (because crossings can make drawings hard to read) or for road networks or polyhedra where they naturally form

Some of their main properties (few edges, low degeneracy, low treewidth) can help in finding fast algorithms for these graphs

Many problems have natural duals (e.g. Euler tour $\Leftrightarrow$ 2-coloring), can be used to transform problems into a different form that might be more recognizable

Duality of spanning trees $\Rightarrow$ Euler's formula $\Rightarrow$ number of edges

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