# A New Bounding Scheme for Influence Diagrams

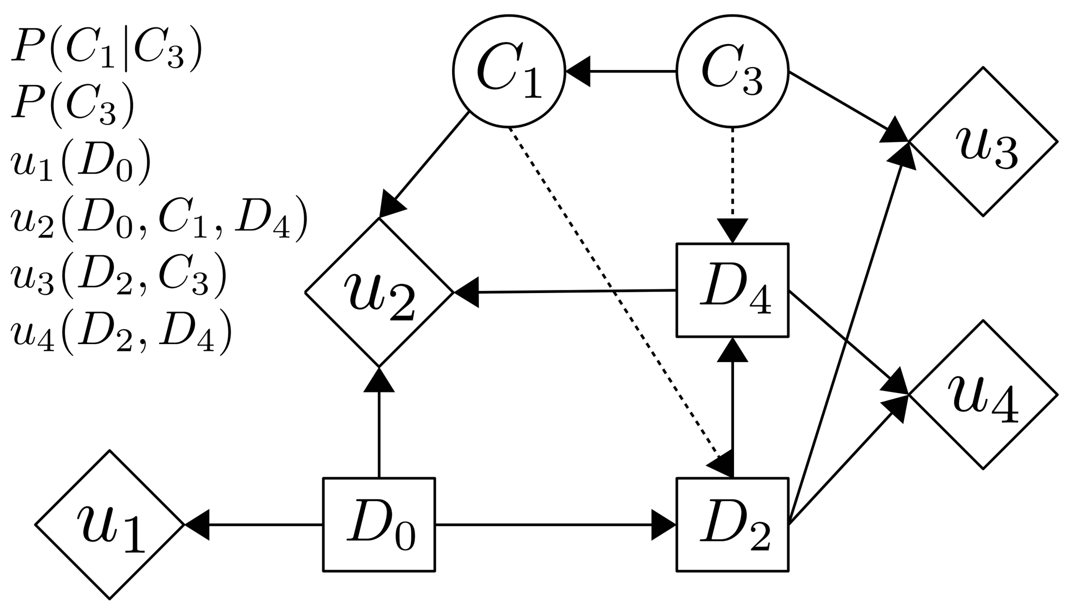
Radu Marinescu (IBM Research), Junkyu Lee (IBM Research), Rina Dechter (UCIrvine)

## Motivation and Contribution

- IDs are a powerful formalism for reasoning with sequential decision-making problems under uncertainties
  - Involve random (or chance) variables, decision variables and utility functions
- Task: find the maximum expected utility (MEU) and the corresponding optimal policy
  - Notoriously difficult to solve exactly in practice
- Recent work focused on bounding the MEU
  - E.g., information relaxation, reformulation to Marginal MAP, partitioning over join-trees
- Contribution:
  - Revisit multi-operator cluster DAG (MCDAG) decompositions for influence diagrams
  - Partitioning-based (mini-bucket) approximation for MCDAGs to upper bound the MEU
  - Apply cost-shifting to tighten the upper bounds further
  - In practice, we obtain significantly tighter bounds (by several orders of magnitude) than existing schemes

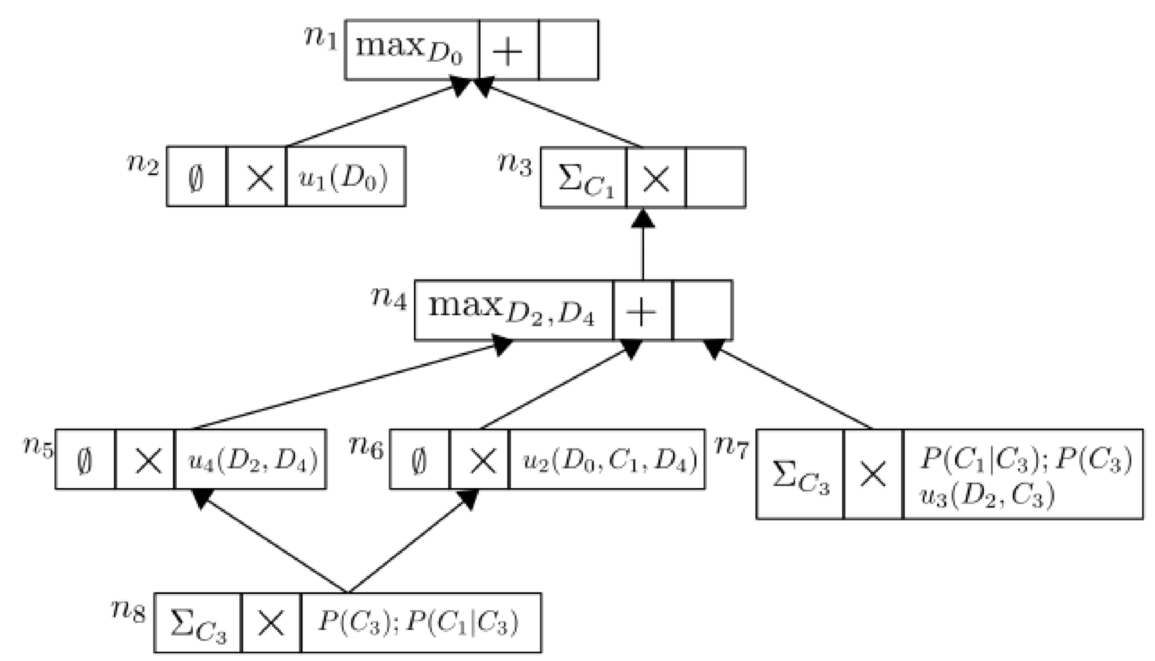
## Influence Diagrams

$P(C_1|C_3)$
$P(C_3)$
$u_1(D_0)$
$u_2(D_0, C_1, D_4)$
$u_3(D_2, C_3)$
$u_4(D_2, D_4)$




$MEU = \max_{D_0} \sum_{C_1} \max_{D_2} \sum_{C_3} \max_{D_4} P(C_1|C_3) \cdot P(C_3) \cdot (u_1(D_0) + u_2(D_0, C_1, D_4) + u_3(D_2, C_3) + u_4(D_2, D_4))$

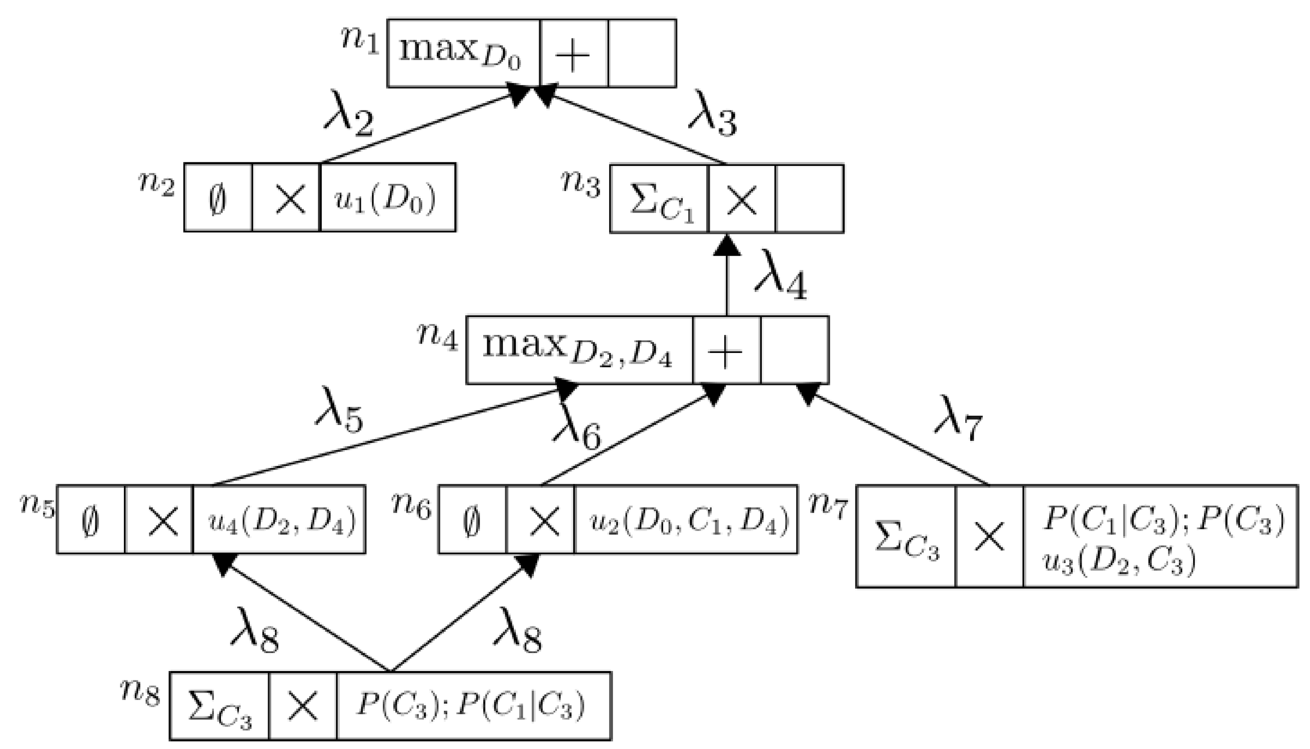
## MCDAG




$V(n_4) = \{D_2, D_4\}; \Psi(n_4) = \emptyset; ch(n_4) = \{n_5, n_6, n_7\}; \oplus = max; \otimes = +$

## Variable Elimination

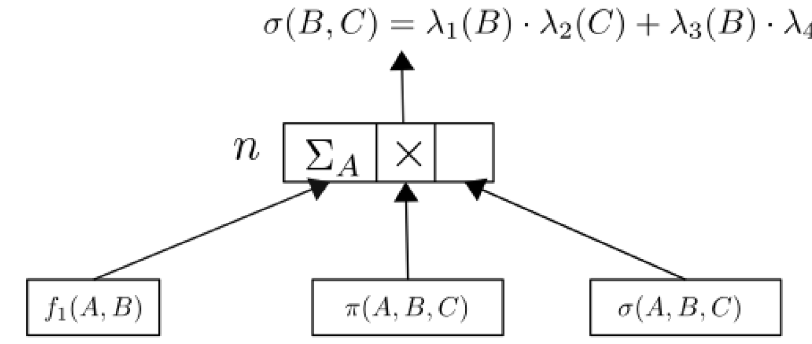
- Compute the MEU via message passing over the MCDAG, bottom-up from leaves to the root:




$$MEU = \max_{D_0} \lambda_2 + \lambda_3$$
$$\lambda_2 = u_1(D_0)$$
$$\lambda_3 = \sum_{C_1} \lambda_4$$
$$\lambda_4 = \max_{D_2} \max_{D_4} (\lambda_5 + \lambda_6 + \lambda_7)$$
$$\lambda_5 = \lambda_8 \cdot u_4(D_2, D_4)$$
$$\lambda_6 = \lambda_8 \cdot u_2(D_0, C_1, D_4)$$
$$\lambda_7 = \sum_{C_3} P(C_3) \cdot P(C_1|C_3) \cdot u_3(D_2, C_3)$$
$$\lambda_8 = \sum_{C_3} P(C_3) \cdot P(C_1|C_3)$$

## Mini-bucket approximation for SUM and MAX Clusters

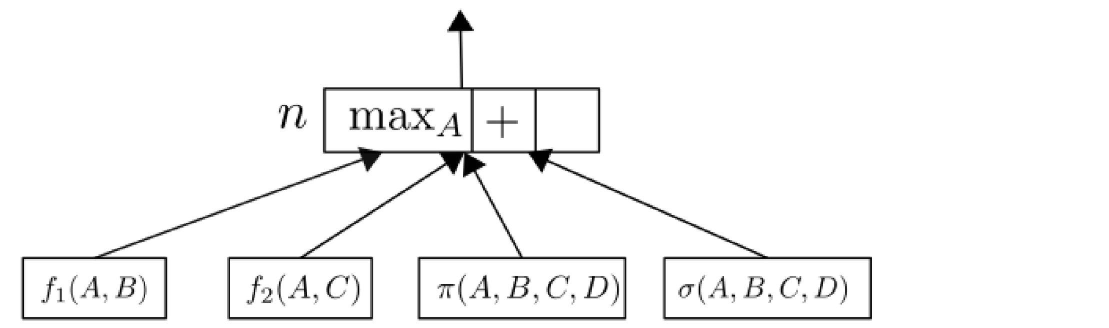
$\sigma(B, C) = \lambda_1(B) \cdot \lambda_2(C) + \lambda_3(B) \cdot \lambda_4(C)$




$\pi(A, B, C) = h_1(A, B) \cdot h_2(A, C)$
$\sigma(A, B, C) = g_1(A, B) \cdot g_2(A, C) + g_3(A, B) \cdot g_4(A, C)$

$$\lambda = \sum_A f_1(A,B) \cdot h_1(A,B) \cdot g_1(A,B) \cdot h_2(A,C) \cdot g_2(A,C) + \sum_A f_1(A,B) \cdot h_1(A,B) \cdot g_3(A,B) \cdot h_2(A,C) \cdot g_4(A,C)$$
$$\leq (\sum_A f_1(A,B) \cdot h_1(A,B) \cdot g_1(A,B)) \cdot (\sum_A h_2(A,C) \cdot g_2(A,C))$$
$$+ (\sum_A f_1(A,B) \cdot h_1(A,B) \cdot g_3(A,B)) \cdot (\sum_A h_2(A,C) \cdot g_4(A,C))$$
$$= \lambda_1(B) \cdot \lambda_2(C) + \lambda_3(B) \cdot \lambda_4(C)$$

$\sigma(B, C, D) = \lambda_1^1(B, C) + h_2(C, D) \cdot \lambda_2^1(B) + \lambda_3^1(B, C) + g_4(C, D) \cdot \lambda_3^2(B)$



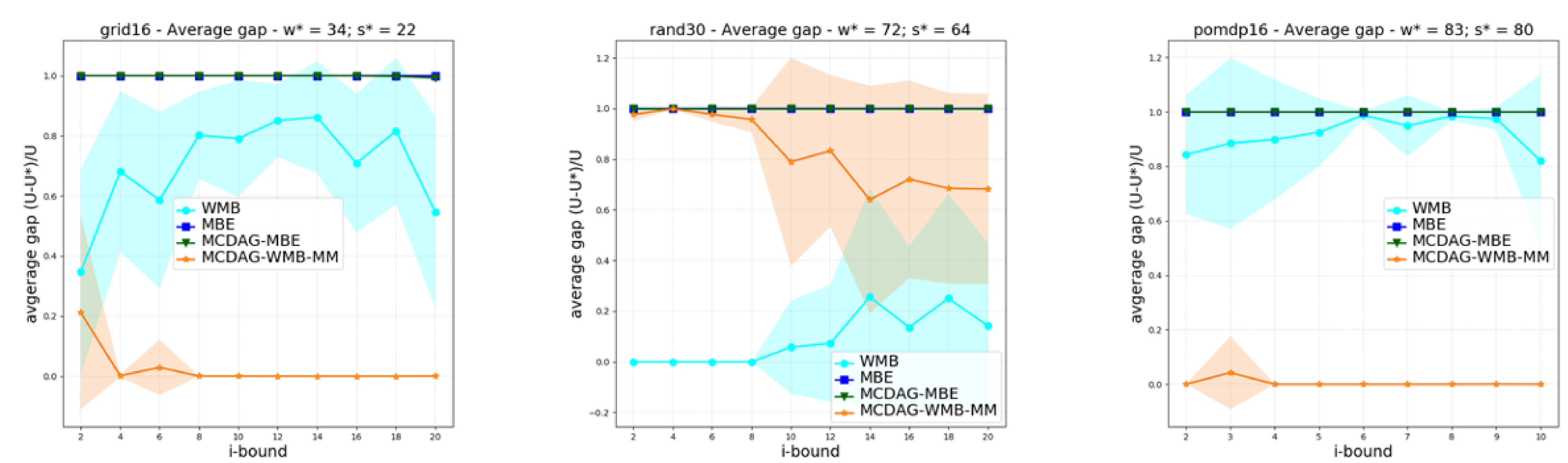

$\pi(A, B, C, D) = h_1(A, B) \cdot h_2(C, D)$
$\sigma(A, B, C, D) = g_1(A, B) \cdot g_2(A, C) + g_3(A, B) \cdot g_4(C, D)$

$$\lambda = \max_A f_1(A,B) + f_2(A,C) + h_1(A,B) \cdot h_2(C,D) + g_1(A,B) \cdot g_2(A,C) + g_3(A,B) \cdot g_4(C,D)$$
$$\leq \max_A (f_1(A,B) + f_2(A,C)) + \max_A h_1(A,B) \cdot h_2(C,D)$$
$$+ \max_A g_1(A,B) \cdot g_2(A,C) + \max_A g_1(A,B) \cdot g_4(C,D)$$
$$= \lambda_1^1(B,C) + \lambda_2^1(B) \cdot h_2(C,D) + \lambda_3^1(B,C) + \lambda_4^1(B) \cdot g_4(C,D)$$

## Cost-shifting via moment-matching

- Use weighted elimination for variable $X$: $\Sigma_X^w f = (\Sigma_X f^{\frac{1}{w}})^w$
- Moment-matching between mini-buckets for SUM clusters (when eliminating variable $X$)
  - Let $Q = \{Q_1, \dots, Q_R\}$ be a mini-bucket partitioning such that $\psi_r = \prod_{f \in Q_r} f$ and $w_r > 0$ such that $\sum_r w_r = 1$
  - Re-parameterize $\psi_r = \psi_r(\frac{\mu}{\mu_r})^{w_r}$, $\mu_r = \Sigma_{Y_r} \psi_r^{1/w_r}$, $\mu = \prod_r \mu_r^{w_r}$, $Y_r = vars(Q_r) \setminus X$
- Moment-matching between mini-buckets for MAX clusters (when eliminating variable $X$)
  - If $\psi_r = \prod_{f \in Q_r} f$ then re-parameterize $\psi_r = \psi_r(\frac{\mu}{\mu_r})$, $\mu_r = \max_{Y_r} \psi_r$, $\mu = (\prod_r \mu_r)^{1/R}$, $Y_r = vars(Q_r) \setminus X$
  - If $\psi_r = \sum_{f \in Q_r} f$ then re-parameterize $\psi_r = \psi_r - \mu_r + \frac{1}{R}\mu$, $\mu_r = \max_{Y_r} \psi_r$, $\mu = \sum_r \mu_r$, $Y_r = vars(Q_r) \setminus X$

## Experiments




Gap $\rho = \frac{(U - U^*)}{U}$, relative to the tightest upper bound $U^*$

$w^*$- induced width (join-tree); $s^*$- induced width (MCDAG)

## Conclusion

- Revisit MCDAGs for IDss and develop a mini-bucket approximation scheme for bounding the MEU
- MCDAGs are more sensitive to the underlying problem structure than strong join-trees (i.e., smaller induced-widths)
- Apply cost-shifting by moment-matching to tighten the bounds further
- Experiments on difficult benchmark problem instances demonstrate the effectiveness of our proposed bounding scheme compared with existing state-of-the-art approaches