Probabilistic Inference Modulo Theories

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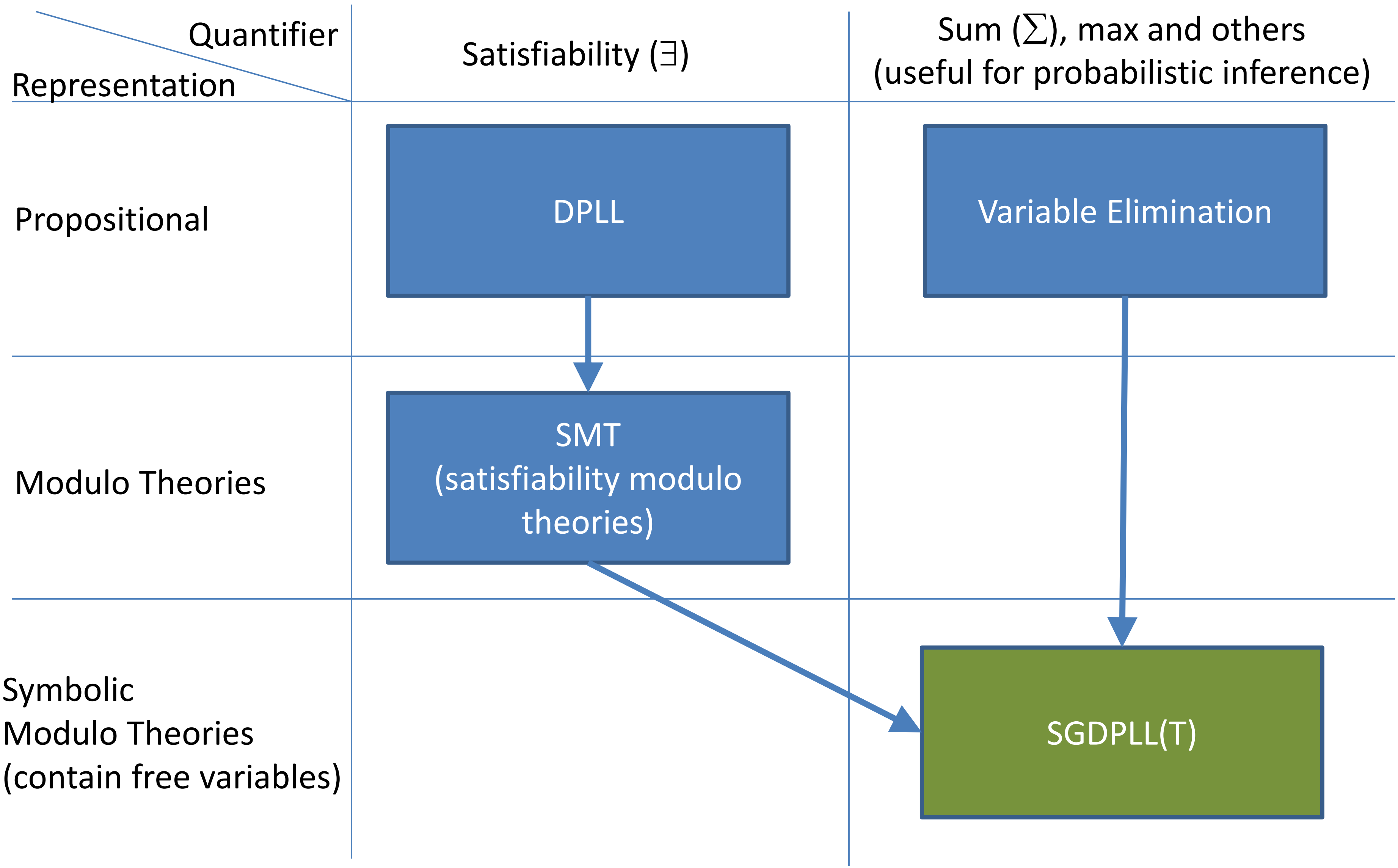
Overview



Quantifier
Representation
Satisfiability (∃)
Sum (∑), max and others
(useful for probabilistic inference)
Propositional
DPLL
Variable Elimination
Modulo Theories
SMT
(satisfiability modulo theories)
Symbolic
Modulo Theories
(contain free variables)
SGDPLL(T)

Motivation

- Consider a probabilistic model on string-valued variables:

```
P(announcement | title, speaker, venue, abstract) =
  if announcement = title + speaker + venue + abstract
     then 0.7 else
  if announcement = title + venue + speaker + abstract
     then 0.3 else 0

P(speaker | name) =
  if speaker = "Prof." + name
     then 0.1 else
  if speaker = name then 0.9 else 0

... // more statements, defining knowledge about
    // names, titles etc.
```

- Exact graphical models algorithms typically iterate over values of each variable, but here they are infinite
- Sampling has its own set of disadvantages

Probabilistic inference with Integers (polynomials and inequalities)

- Consider the following model:

$P(z \in 1..1000) \quad \propto \quad z^2$

$P(x \in 1..1000 \mid z) \propto \quad \text{if } x = z \text{ then } x \text{ else } 0.6$

$P(y \in 1..1000 \mid x) \propto \quad \text{if } x > y \wedge y \neq 5 \text{ then } x^2 - y \text{ else } 0.9$

$P(x, y, z) \propto \quad z^2 \times (\text{if } x > y \wedge y \neq 5 \text{ then } x^2 - y \text{ else } 0.9) \times (\text{if } x = z \text{ then } x \text{ else } 0.6)$

Marginal $P(y) = \sum_{z,x} P(x, y, z)$

$$\propto \sum_z z^2 \sum_x \begin{array}{c}(\text{if } x > y \wedge y \neq 5 \text{ then } x^2 - y \text{ else } 0.9) \\ \times (\text{if } x = z \text{ then } x \text{ else } 0.6)\end{array}$$

- How can we compute this sum <u>without iterating over all the values</u>?

Example

Background - Satisfiability

- We want to compute

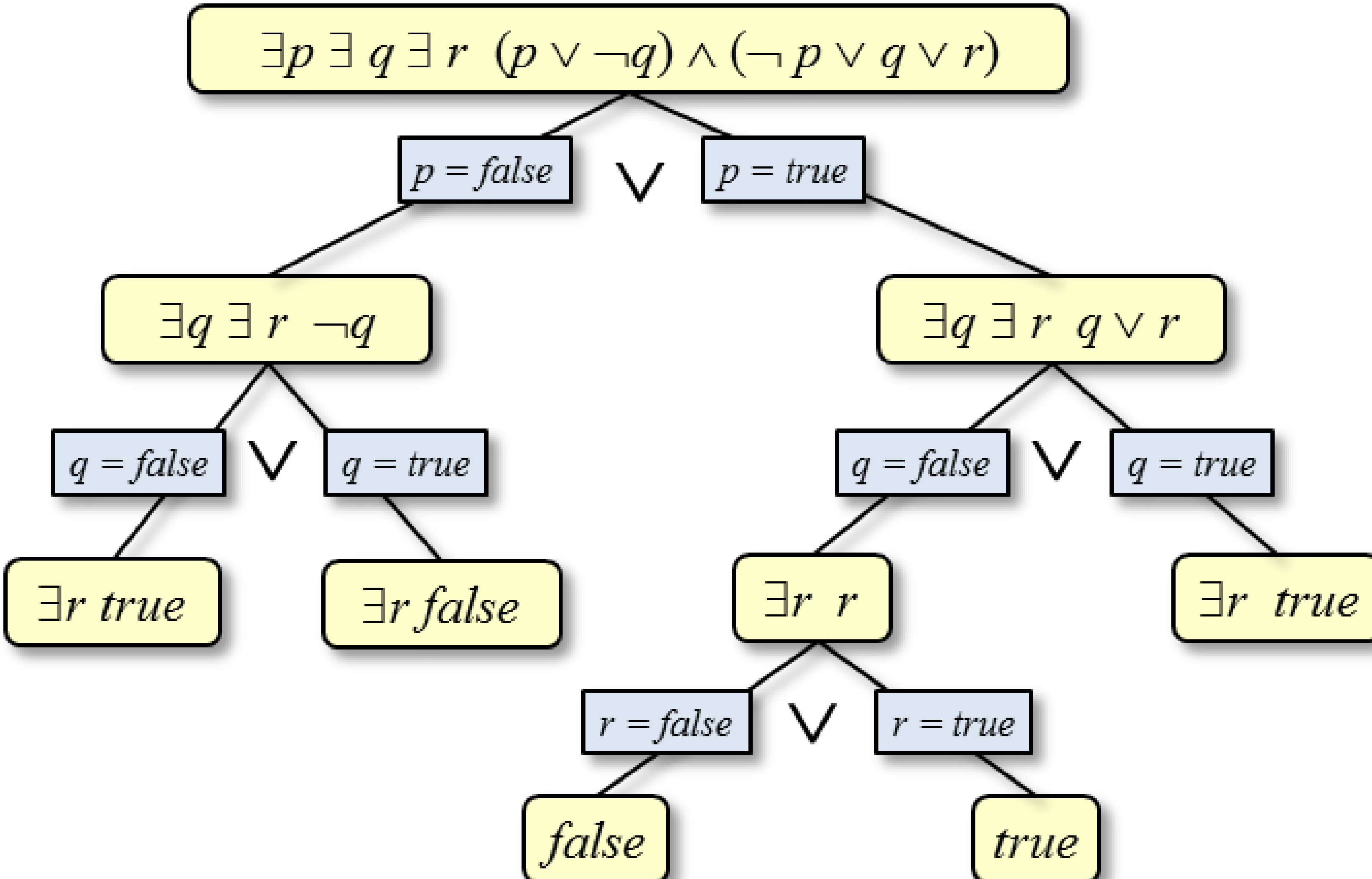
$$\sum_x (\text{if } x > y \wedge y \neq 5 \text{ then } x^2 - y \text{ else } 0.9) \times (\text{if } x = z \text{ then } x \text{ else } 0.6)$$

- The Davis-Putnam-Logemann-Loveland (DPLL) algorithm solves the problem of *satisfiability:*

$$\exists p \, \exists \, q \, \exists \, r \;\; (p \vee \neg q) \wedge (\neg \, p \vee q \vee r)$$

- This is similar to what we need, but for
  - Existential quantification instead of summation
  - Propositional variables (no theories)
  - Total quantification (no free variables)

Background - DPLL



Background – Satisfiability Modulo Theories (SMT)

- Satisfiability modulo theories generalizes satisfiability to non-propositional logic (includes arithmetic, inequalities, lists, uninterpreted functions, and others)

$$\exists x\ \exists y \exists z\ \exists l (x \neq 5y \vee y > z) \wedge cons(x, nil) \neq cdr(l)$$

- This is closer to what we need (since it works on theories), but for
  - Existential quantification instead of summation
  - Total quantification (no free variables)

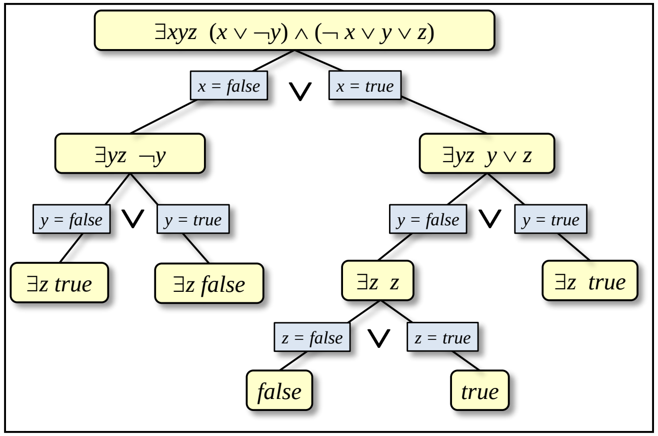
First Contribution: Symbolic Generalized DPLL(T)

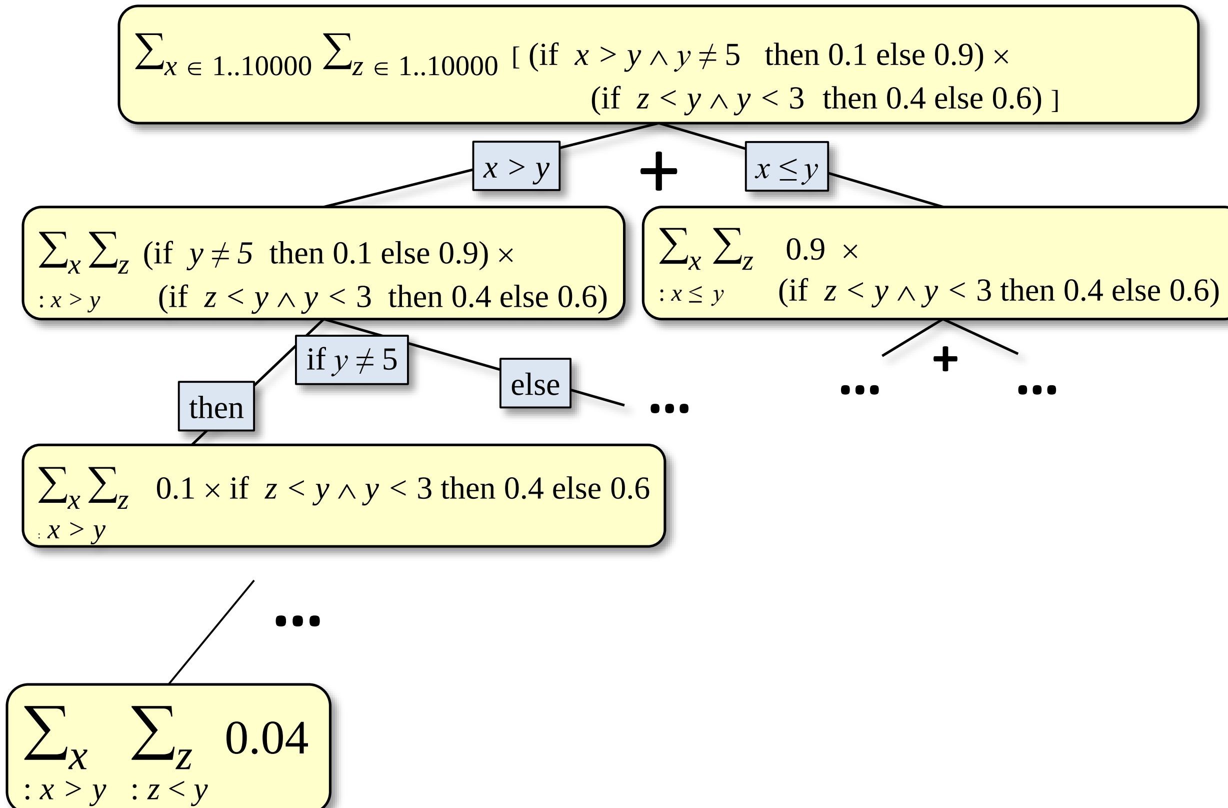
- Similar to SMT, but based on
  - Summation (or other quantifiers), besides $\exists$
  - Partial quantification (free variables)

$$\sum_{x \in 1..10000} \sum_{z \in 1..10000} (\text{if } x > y \wedge y \neq 5 \text{ then } 0.1 \text{ else } 0.9) \times (\text{if } z < y \wedge y < 3 \text{ then } 0.4 \text{ else } 0.6)$$

- Note that *y* is a free variable
- Summed expression is not Boolean
- Language is not propositional (≠, <, …)

Symbolic Generalized DPLL(T) – SGDPLL(T)

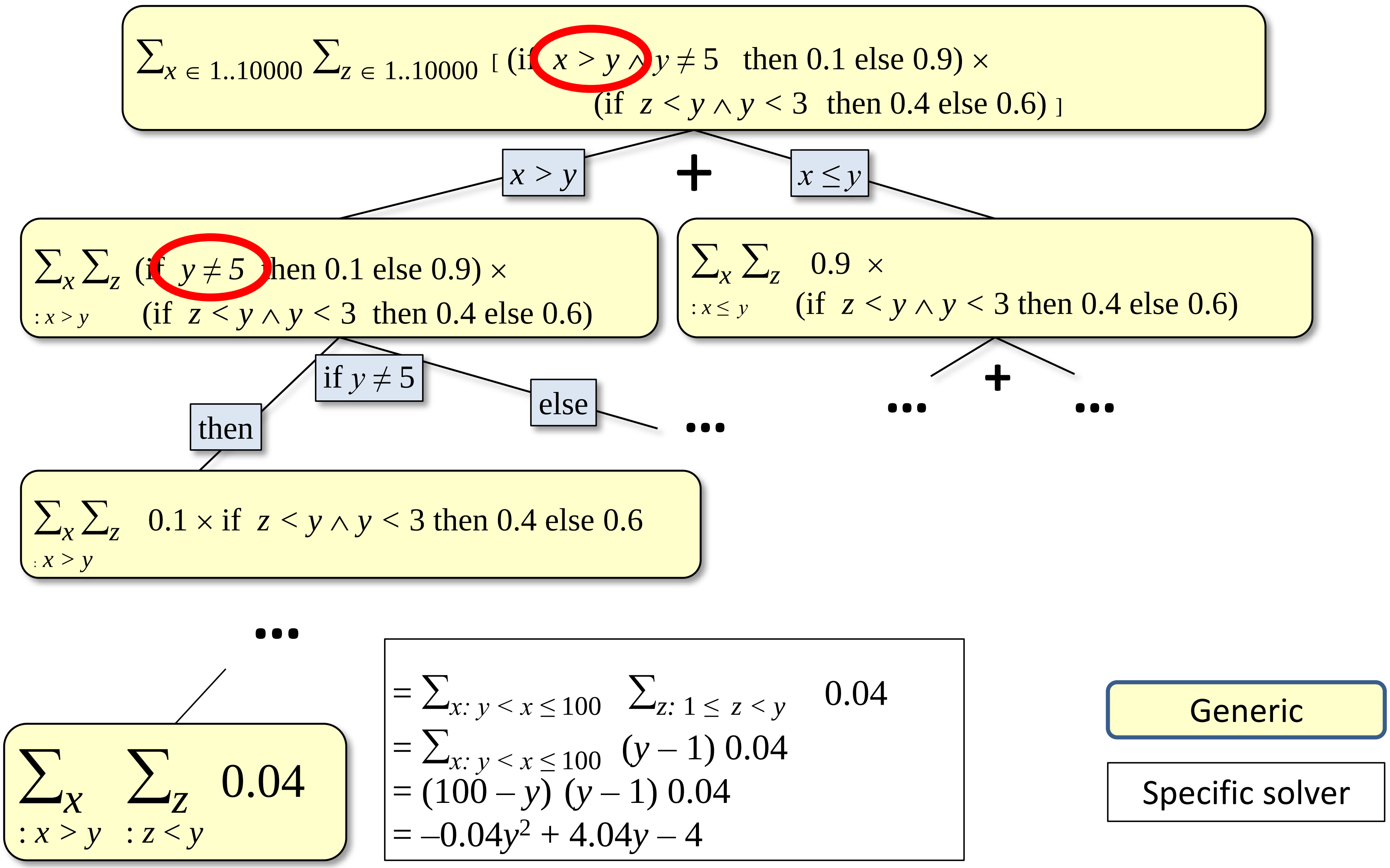




Condition on literals until base case with no literals in main expression:

$$= \sum_{x:\, y < x \le 100} \sum_{z:\, 1 \le z < y} 0.04$$
$$= \sum_{x:\, y < x \le 100} (y - 1)\, 0.04$$
$$= (100 - y)\,(y - 1)\, 0.04$$
$$= -0.04y^2 + 4.04y - 4$$

Symbolic Generalized DPLL(T)



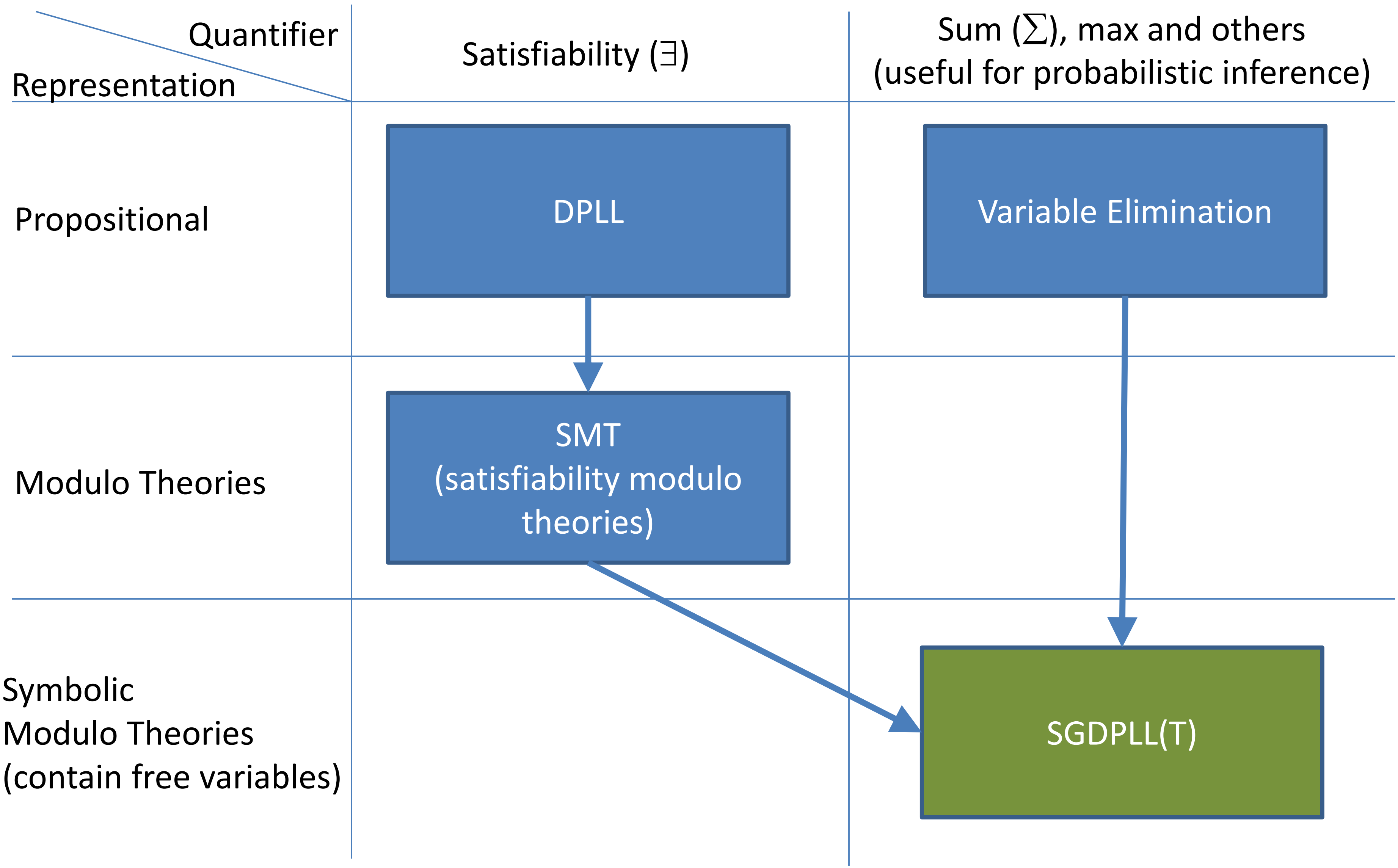
Second Contribution: Solver for summation with difference arithmetic on bounded integers theory on polynomials

- $\sum_{z:1 \leq z < y} 0.04$
  is an easy case:
  - Constant body expression
  - Single lower bound, single upper bound, no ≠
- $\sum_{z:\ 1 \leq z \ \wedge\ x \leq z \ \wedge\ z \neq 5 \ \wedge\ z < y} z^2 - 2z$
  is more complicated:
  - Requires splitting on $x < 1$ to decide which is lower bound
  - Requires splitting on $5 < y$ to decide if $z \neq 5$ is relevant
  - Requires a generalized *Faulhaber's formula* to sum over polynomial
- This splitting needs to be carefully implemented (simplifying at every split is too expensive)

Since paper's final version…

- … linear real arithmetic added as a separate theory
- Theories are automatically combined, so now we can define hybrid models on discrete and continuous variables and solve them symbolically

Overview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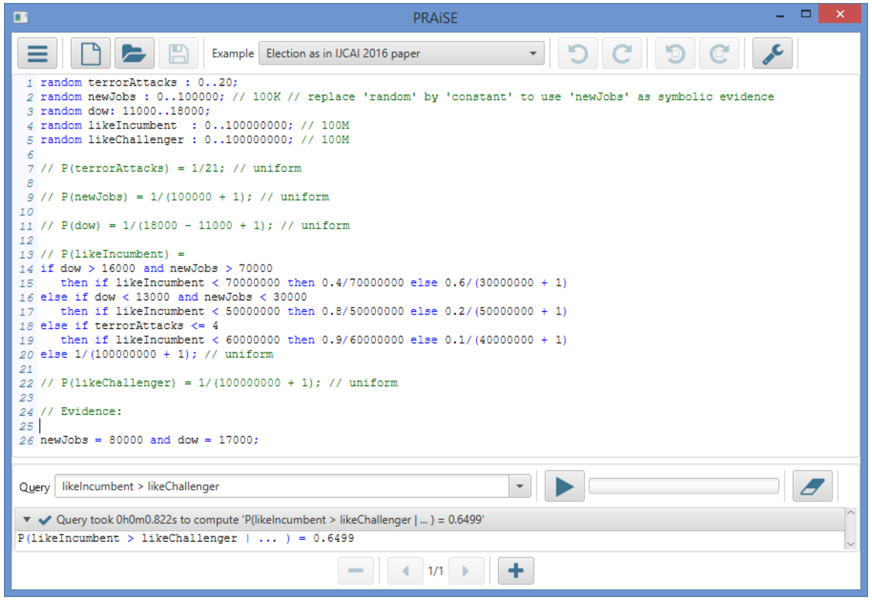
Relation to Lifted Inference

- Lifted First-order Probabilistic Inference (Poole 2003, de Salvo Braz 2005) performs probabilistic inference on first-order predicates, without iterating over all values of their arguments

  $\forall X \;\; P(\, cancer(X) \;\; | \; smoker(X)\,) = 0.6$
  $P(\, smoker(mary)\,) = 0.01$

- In the SMT vocabulary, that can be seen as a theory solver for uninterpreted functions
- This paper can be seen as lifted inference on interpreted functions
- Traditional lifted inference can be incorporated as a solver for uninterpreted functions in SGDPLL(T)

Proof-of-concept Experiment



- Grounded the elections example into a regular graphical models and using VEC (Gogate & Dechter, 2011)
- Had to decrease domain size to 180 to keep it manageable, and VEC was still 20 times slower

Conclusion

- This is graphical models, but defined with richer representations (theories)
- Similar to SMT, but with summation and free variables
- Symbolic: result is a math expression on free variables
- Future work
  - solvers for more theories
  - unit propagation, clause learning from SAT literature
  - bounded approximations for limiting the search

Thank you!