ICS 6A
Solution to Homework Assignment 7
Winter 2004

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Answer the following questions (explain your answers).

1. Rosen, page 253, problem 3.

   Please see *"Solutions to Odd-Numbered Exercises"* of Rosen.(Page S-24)

2. Rosen, page 253, problem 10.
   Proof: Let P(n) be "$1 \cdot 1! + 2 \cdot 2! + \cdot + n \cdot n! = (n+1)! - 1$", where $n = 1, 2, 3, \cdots$

   - Basis step: for n = 1, $1 \cdot 1! = 1 = (1+1)! - 1 \Rightarrow$ P(1) is true.
   - Inductive step: Assume P(n) is true, i.e. $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$, then:
     $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)!$
     $= (n+1)! - 1 + (n+1) \cdot (n+1)!$ From induction hypothesis
     $= (n+1)! \cdot (1 + n + 1) - 1$
     $= (n+1)! \cdot (n+2) - 1$
     $= (n+2)! - 1$

   The last equation shows that P(n+1) is true. This completes the inductive step and completes the proof.

3. Rosen, page 253, problem 13.
   Please see *"Solutions to Odd-Numbered Exercises"* of Rosen. (Page S-24)

4. Rosen, page 253, problem 16.
   Proof: Let P(n) be "$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$"

   - Basis step: $1 \cdot 2 \cdot 3 = 6 = 1 \cdot (1+1) \cdot (1+2) \cdot (1+3)/4 \Rightarrow$ P(1) is true.
   - Inductive step: Assume P(n) is true, i.e. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$, then:
     $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) + (n+1)(n+2)(n+3)$
     $= n(n+1)(n+2)(n+3)/4 + (n+1)(n+2)(n+3)$ From induction hypothesis
     $= (n+1)(n+2)(n+3) \cdot (n/4 + 1)$
     $= (n+1)(n+2)(n+3)(n+4)/4$

   The last equation shows that P(n+1) is true. This completes the inductive step and completes the proof.

5. Rosen, page 253, problem 18.
   Proof: Let P(n) be "$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$", where $n = 2, 3, \cdots$

   - Basis step: for n = 2, $1 + \frac{1}{4} = \frac{5}{4} < 2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4} \Rightarrow$ P(2) is true.
   - Inductive step: Assume P(n) is true, i.e. $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$, then:
     $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}$
     $< 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$ From induction hypothesis
     $= 2 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n} + \frac{1}{(n+1)^2}$
     $= 2 - \frac{1}{n+1} + \frac{n(n+1) - (n+1)^2 + n}{n(n+1)^2}$
     $= 2 - \frac{1}{n+1} + \frac{n^2 + n - (n^2 + 2n + 1) + n}{n(n+1)^2}$
     $= 2 - \frac{1}{n+1} + \frac{n^2 + n - n^2 - 2n - 1 + n}{n(n+1)^2}$

$= 2 - \frac{1}{n+1} + \frac{-1}{n(n+1)^2}$
$< 2 - \frac{1}{n+1}$ whenever $n > 1$, because $\frac{-1}{n(n+1)^2} < 0$

The last inequlity shows that P(n+1) is true. This completes the inductive step and completes the proof.

6. Rosen, page 254, problem 22.
Proof: Let P(n) be "6 divides $n^3 - n$", where $n = 0, 1, 2, \cdots$

- Basis step: for n = 0, 6 devides $0^3 - 0 = 0 \Rightarrow$ P(0) is true.
- Inductive step: Assume P(n) is true, i.e. 6 divides $n^3 - n$, then:
  $(n+1)^3 - (n+1)$
  $= n^3 + 3n^2 + 3n + 1 - n - 1$
  $= (n^3 - n) + 3n^2 + 3n$
  $= (n^3 - n) + 3n(n+1)$
  $(n^3 - n)$ can be divided by 6 from induction hypothesis, either n or $n+1$ are even number, then $n(n+1)$ can be devided by 2, so $3n(n+1)$ can be devided by 6, and $(n^3 - n) + 3n(n+1)$ can be devided by 6. This means that P(n+1) is true.

This completes the inductive step and completes the proof.

7. Rosen, page 255, problem 45.
Please see the *"Solutions to Odd-Numbered Exercises"* of Rosen.(Page S-26

8. Rosen, page 255, problem 48.
Proof: Let P(n) be "$\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$"

- Basis step: $\neg p_1$ is equivalent to $\neg p_1 \Rightarrow$ P(2) is true.
- Inductive step: Assume P(n) is true, i.e. $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$, then:
  $\neg(p_1 \vee p_2 \vee \cdots \vee p_n \vee p_{n+1}) \Leftrightarrow \neg[(p_1 \vee p_2 \vee \cdots \vee p_n) \vee p_{n+1}]$
  is equivalent to $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \wedge \neg p_{n+1}$ by "De Morgan's laws"
  is equivalent to $(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n) \wedge \neg p_{n+1}$ From induction hypothesis
  is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}$

The last "equivalent relation" shows that P(n+1) is true. This completes the inductive step and completes the proof.

9. Rosen, page 236, problem 3.

a) $2^n + 1$: $a_0 = 2^0 + 1 = 2$
$a_1 = 2^1 + 1 = 3$
$a_2 = 2^2 + 1 = 5$
$a_3 = 2^3 + 1 = 9$

b) $(n+1)^{n+1}$: $a_0 = (0+1)^{0+1} = 1$
$a_1 = (1+1)^{1+1} = 4$
$a_2 = (2+1)^{2+1} = 27$
$a_3 = (3+1)^{3+1} = 256$

c) $\lfloor n/2 \rfloor$: $a_0 = \lfloor 0/2 \rfloor = 0$
$a_1 = \lfloor 1/2 \rfloor = 0$
$a_2 = \lfloor 2/2 \rfloor = 1$
$a_3 = \lfloor 3/2 \rfloor = 1$

d) $\lfloor n/2 \rfloor + \lceil n/2 \rceil$: $a_0 = \lfloor 0/2 \rfloor + \lceil 0/2 \rceil = 0 + 0 = 0$
$a_1 = \lfloor 1/2 \rfloor + \lceil 1/2 \rceil = 0 + 1 = 1$
$a_2 = \lfloor 2/2 \rfloor + \lceil 2/2 \rceil = 1 + 1 = 2$
$a_3 = \lfloor 3/2 \rfloor + \lceil 3/2 \rceil = 1 + 2 = 3$

10. Rosen, page 236 problem 8.
Answer: The terms could be odd numbers greater than 1;
the terms could be prime numbers greater than 2;
the terms could be odd numbers not divisible by 9;
the terms could numbers greater than 2 and not divisible by 4 and 6; $\cdots$
There are infinitely many other possibilities.

11. Rosen, page 236 problem 10.
a) $a_n = n^2 + 2 \cdot n + 3$, where $n = 0, 1, 2, \cdots$
b) $a_n = 4 \cdot n + 7$, where $n = 0, 1, 2, \cdots$
c) 10^n followed by the sum of 10^n and previous terms respectively, where $n = 0, 1, 2, \cdots$.
d) The Fibonacci sequence $f(n+1)$ listed $2n - 1$ times.
e) $3^n - 1$, where $n = 0, 1, 2, \cdots$
f) $\frac{(2n+1)!}{2^n \cdot n!}$, where $n = 0, 1, 2, \cdots$
g) One 1 followed by two 0s, three 1s, four 0s, and so on.
h) 2^{2^n}, where $n = 0, 1, 2, \cdots$

12. Rosen, page A-3, problem 2.
a) $\log_2 1024 = \log_2 2^{10} = 10$
b) $\log_2 \frac{1}{4} = \log_2 2^{-2} = -2$
c) $\log_4 8 = \frac{\log_2 8}{\log_2 4}$ **THEOREM 3** on page A-2 of Rosen.
$= \frac{\log_2 2^3}{\log_2 2^2} = \frac{3}{2}$

13. Rosen, page A-3, problem 4.
Proof: Using **THEOREM 2** rule 2: $\log_b LeftHandSide = \log_b a^{\log_b c} = \log_b c \cdot \log_b a$
$\log_b RightHandSide = \log_b c^{\log_b a} = \log_b a \cdot \log_b c$
$\log_b LeftHandSide = \log_b RightHandSide \Rightarrow LeftHandSide = RightHandSide$
So, $a^{log_b c} = c^{\log_b a}$