Solution to Sample Midterm
Winter 2004

NOTE: You should always give the number as the final answer like 1 a), not just C(14, 4) like the answers of the rest of the problems.

1. Suppose I give 10 \$1 bills to five kids. The dollar bills are not distinct but the kids are distinct.

   (a) How many ways are there to distribute the money? **Answer:** $C(10+5-1,10) = C(14,10) = C(14,4) = \frac{14!}{4!\cdot(14-4)!} = 1001$

   (b) How many ways are there to distribute the money so that each kid gets at least one dollar? **Answer:** $C(10-5+5-1, 10-5) = C(9,5) = C(9,4)$

   (c) How many ways are there to distribute the money so that at least one of the kids gets no money? **Answer:** $(a) - (b) = C(10+5-1,10) - C(10-5+5-1,10-5) = C(14,4) - C(9,4)$

2. Suppose that ten people arrive at a party on a rainy night and each person arrives with a distinct umbrella. At the end of the party, each person picks up one umbrella (not necessarily their own) before they leave.

   (a) How many ways are there for umbrellas to end up with people? **Answer:** $10!$

   (b) How many ways are there for the umbrellas to be picked up by people so that one paritcular guest, George, gets his original umbrella? **Answer:** $(10-1)! = 9!$

   (c) Now suppose that the ten guests consist of a family of four and a family of six. The family of four picks up four umbrellas and the family of six picks up six umbrellas. Since everyone in each family leaves together, we only care about which umbrellas end up with which family. How many ways are there for the umbrellas to be picked up? **Answer:** $C(10,4)\cdot C(10-4,6) = C(10,4)$, which is equivalent with the problem "Divide 10 umbrellas into 2 groups with 4 and 6 umbrellas each.".

   (d) Answer the previous question if there is a family of three, a family of five and one couple. **Answer:** $C(10,3)\cdot C(10-3,5)\cdot C(10-3-5,2) = C(10,3)\cdot C(7,5) = C(10,3)\cdot C(7,2)$, which is equivalent with the problem "Divide 10 umbrellas into 3 groups with 3, 5 and 2 umbrellas each.".

3. Let $p(x)$ and $q(x)$ denote the following open statements:

$$\begin{aligned} p(x):&\quad x^2 - 8x + 15 = 0 \\ q(x):&\quad x \text{ is odd} \end{aligned}$$

   For the universe of all integers, determine the truth or falsity of each of the following statements. Briefly justify your answer. In particular, if a statement is false, give a counter-example.
   Answer: $p(x) \rightarrow x = 3$ *or* $x = 5$, 3 and 5 are roots of p(x).

   (a) $\forall x[p(x) \rightarrow q(x)]$: True. $x = 3$ *or* $x = 5 \rightarrow x$ *is odd*

(c) $\forall x[q(x) \to p(x)]$: False. eg: $x = 1(odd) \to -p(x)$

(d) $\exists x[q(x) \to p(x)]$. True. eg: $x = 3 \ or \ x = 5 \to x \ is \ odd$

4. Let $A = \{x \in Integers | x \leq 4\}$ and let $B = \{x \in Integers | 3 < x \leq 8\}$. Determine each of the following sets:

a) $A \cap B$	**b)** $A \cup B$	**c)** $\overline{A}$
d) $A \triangle B$	**e)** $A - B$	**f)** $B - A$

Where $A \triangle B$ is symmetric difference which is equivalent to: $A \triangle B = (A - B) \cup (B - A)$

Answer: A = {⋯, -3, -2, -1, 0, 1, 2, 3, 4}, B = {4, 5, 6, 7, 8}
a) $A \cap B = \{4\}$
b) $A \cup B =$ {⋯, -3, -2, -1, 0, 1, 2, ⋯, 8} $= \{x \in Integers | x \leq 8\}$
c) $\overline{A} =$ {5, 6, 7, ⋯} $= \{x \in Integers | x > 4\}$
d) $A \triangle B =$ {⋯, -2, -1, 0, 1, 2, 3, 5, 6, 7, 8} $= \{x \in Integers | x \leq 8 \ and \ x \neq 4\}$
e) $A - B =$ {⋯, -2, -1, 0, 1, 2, 3} $= \{x \in Integers | x \leq 3\}$
f) $B - A =$ {5, 6, 7, 8} $= \{x \in Integers | 4 < x \leq 8\}$

5. (a) What is the probability that a card selected from a deck is an ace or a heart. **Answer:** $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

   (b) What is the probability that a die never comes up an even number when it is rolled six times. **Answer:** $(\frac{1}{2})^6 = \frac{1}{64}$

   (c) What is the expected number of times that a die comes odd if it is rolled 6 times. **Answer:** $6 \times \frac{1}{2} = 3$

6. Fine the probability that a family with 4 children does not have a boy if the sexes of the children are independent and if

   (a) A boy and a girl are equaly likely. **Answer:** $(\frac{1}{2})^4 = \frac{1}{16}$

   (b) The probability of a boy is 0.51. **Answer:** $(1 - 0.51)^4 = 0.0576$