ICS 6A
Sample Final
Winter 2004
Instructor: Rina Dechter

1. *(15 points)* Solve the following recurrence relation:

$$a_n = 2na_{n-1}, a_0 = 1.$$

   Express your answer as a closed form solution which is a function of n.

2. *(20 points)* Consider the recurrence relation, defined by

$$\begin{aligned} p_1 &= 0 \\ p_n &= 3n \cdot p_{n-1} + n \quad \text{for } n > 1 \end{aligned}$$

   Prove by induction that for $n \geq 0$,

$$p_n = \sum_{i=1}^{n-1} \frac{3^{n-i-1} n!}{i!}$$

   .

3. (20 points) Show by induction that

$$\sum_{i=1}^{n} i(i+1)/2 = n(n+1)(n+2)/6$$

4. *(15 points)* Solve the following recurrence relation:

$$a_0 = 1$$
$$a_1 = 9$$
$$a_n = 6 \cdot a_{n-1} - 9 \cdot a_{n-2}, \quad \textit{for } n > 1$$

   Express your answer as a closed form solution which is a function of n.

5. (15 points) Solve the following recurrence relation: $a_n = 3a_{n-1} + 2$, $a_1 = 1$. Express your answer as a closed for solution which is a function of n. (Try to prove your solution by induction.)

6. (10 points) How many sequences of five letters are there in which exactly two are vowels. "Y" does not count as a vowel and we are not distinguishing between lower and upper case.

7. (15 points) I have a 100 comic books which I am dispensing at random to ten kids. That is, each comic book is given to one of the ten kids chosen at random. Note that kids are distinct and the comic books are distinct.

(a) (5 points) What is the probability that a specific kid (say George) gets no comic book at all?

(b) (10 points) If the books are indistinguishable, and the kids are distinguishable, what is the probability that there is some kid who gets no comic book?

(c) (15 points) Still suppose that the books and the kids are all distinguishable, what is the expected number of comic books that George gets? (Full credit for a numeric answer).

8. *(35 points)*

   (a) A pair of dice is rolled once. What is the probability that the total number of points on the dice is seven?

   (b) What is the expected number of times we get a total of 7 in one roll?

   (c) Now suppose that a pair of dice is rolled twice. You can assume that the outcomes of the two rolls are independent.

   (d) What is the probability that exactly one of the two rolls comes up seven?

   (e) What is the probability that at least one of the two rolls comes up seven?

   (f) What is the probability that both rolls come up seven, conditioned on the fact that at least one roll comes up seven?

   (g) What is the expected number of times that we get a total of seven in two independent rolls?

9. *(10 points)* The odds in playing a slot machine are as follows: the probability that the player loses his quarter is 9/10. The probability that four quarters comes back is 9/100. The probability of hitting the jackpot (forty quarters) is 1/100. You can assume that the outcome of one game is independent of the outcome of any other game. Suppose that a player sits down and plays the game 100 times. Let X be the random variable which denotes the number of times the player hits the jackpot. Let Y denote the number of times the player loses his quarter. Are X and Y independent? Prove your answer.

10. (30 points)

    (a) Sam has just joined Chocoholics Anonymous in an attempt to quit eating chocolate. He normally eats n M&Ms a day. He decides to go on a program in which every week, he cuts down his daily M&M intake by one half. (That is, if he n M&Ms a day for one week, then he eats $n/2$ M&Ms a day for the next week, etc.) Including the first week when he is eating his usual n M&Ms a day, how many weeks will it take him until he is down to at most one M&M a day?

    (b) Assume now that n is a power of two. How many M&Ms will he have eaten during the whole reduction program? You should include the first week in which he eats his usual n M&Ms a day. You can assume he stops eating M&Ms after the week where he eats only one M&M a day.

    (c) Now, suppose, he opts for a different program in which each day he eats three fewer M&Ms than the day before. You can assume that n is a multiple of three and that the program ends when he is down to 0 M&Ms a day. How many M&Ms does he eat over the course of this program? You should include the first day in which he eats his usual n M&Ms.