ICS 6A
Solution to Sample Final
Winter 2004
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1. *(15 points)* Solve the following recurrence relation:

$$a_n = 2na_{n-1}, a_0 = 1.$$

Express your answer as a closed form solution which is a function of n.
Solution: $a_n = 2^n \cdot n!$

$$\begin{aligned} a_n &= 2na_{n-1} \\ &= 2n(2(n-1)a_{n-2}) \qquad = 2^2(n(n-1)) \cdot a_{n-2} \\ &= 2^2(n(n-1))(2(n-2)a_{n-3}) = 2^3(n(n-1)(n-2)) \cdot a_{n-3} \\ &\vdots \\ &= 2^n(n(n-1)(n-2)\cdots 1) \cdot a_{n-n} \\ &= 2^n \cdot n! \cdot a_0 \\ &= 2^n \cdot n! \end{aligned}$$

2. *(20 points)* Consider the recurrence relation, defined by

$$\begin{aligned} p_1 &= 0 \\ p_n &= 3n \cdot p_{n-1} + n \quad \text{for } n > 1 \end{aligned}$$

Prove by induction that for $n > 0$,

$$p_n = \sum_{i=1}^{n-1} \frac{3^{n-i-1} n!}{i!}$$

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Proof: Let P(n) be "$p_n = \sum_{i=1}^{n-1} \frac{3^{n-i-1}n!}{i!}$", where $n = 1, 2, \cdots$

- Basis step: for n = 1, $\sum_{i=1}^{0} \frac{3^{n-i-1}n!}{i!} = 0 \Rightarrow$ P(1) is true.
- Inductive step: Assume P(n) is true, i.e. $p_n = \sum_{i=1}^{n-1} \frac{3^{n-i-1}n!}{i!} = 3^{n-1}n! \cdot \sum_{i=1}^{n-1} \frac{3^{-i}}{i!} = 3^{n-1}n! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!}$, then
  We want to prove $p_{n+1} = \sum_{i=1}^{n+1-1} \frac{3^{n+1-i-1}(n+1)!}{i!}$
  $= 3^n(n+1)! \cdot \sum_{i=1}^{n} \frac{1}{3^i i!}$
  $= 3^n(n+1)! \cdot (\sum_{i=1}^{n-1} \frac{1}{3^i i!} + \frac{1}{3^n n!})$
  $= 3^n(n+1)! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!} + \frac{3^n(n+1)!}{3^n n!}$
  $= 3^n(n+1)! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!} + (n+1)$

  $p_{n+1} = 3(n+1) \cdot p_{n-1} + (n+1)$
  $= (n+1)(3p_n + 1)$

$= (n+1)(3 \cdot 3^{n-1} n! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!} + 1)$ From the hypothesis of induction
$= (n+1) \cdot 3 \cdot 3^{n-1} n! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!} + (n+1)$
$= 3^n (n+1)! \cdot \sum_{i=1}^{n-1} \frac{1}{3^i i!} + (n+1)$

The last line shows that P(n+1) is true. This completes the inductive step and completes the proof.

3. (20 points) Show by induction that

$$\sum_{i=1}^{n} i(i+1)/2 = n(n+1)(n+2)/6$$

Proof: Let P(n) be "$\sum_{i=1}^{n} i(i+1)/2 = n(n+1)(n+2)/6$", where $n = 1, 2, \cdots$

- Basis step: for n = 1,
  $\sum_{i=1}^{1} i(i+1)/2 = 1(1+1)/2 = 1$, and
  $n(n+1)(n+2)/6 = 1(1+1)(1+2)/6 = 1$,
  $\Rightarrow$ P(1) is true.
- Inductive step: Assume P(n) is true, i.e. $\sum_{i=1}^{n} i(i+1)/2 = n(n+1)(n+2)/6$, then for P(n+1)
  LeftHandSide $= \sum_{i=1}^{(} n+1) i(i+1)/2$
  $= \sum_{i=1}^{n} i(i+1)/2 + (n+1)(n+1+1)/2 = n(n+1)(n+2)/6 + (n+1)(n+2)/2$
  From the hypothesis of induction
  $= (n+1)(n+2)(n+3)/6$
  RightHandSide $= (n+1)(n+2)(n+3)/6$
  $\Rightarrow$ LeftHandSide = RightHandSide

The last equation shows that P(n+1) is true. This completes the inductive step and completes the proof.

4. *(15 points)* Solve the following recurrence relation:

$$a_0 = 1$$

$$a_1 = 9$$

$$a_n = 6 \cdot a_{n-1} - 9 \cdot a_{n-2}, \quad for\ n > 1$$

Express your answer as a closed form solution which is a function of n.
Answer: $a_n = 3^n + 2n \cdot 3^n$
$c_1 = 6$, $c_2 = -9$,
$r^2 - 6r + 9 = 0 \Rightarrow r_0 = 3$
$\Rightarrow a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot n \cdot 3^n$, with the initial conditions: $a_0 = 1$, $a_1 = 9$
$\Rightarrow 1 = \alpha_1 \cdot 3^0 + \alpha_2 \cdot 0 \cdot 3^0 = \alpha_1$ and
$9 = \alpha_1 \cdot 3^1 + \alpha_2 \cdot 1 \cdot 3^1 = 3\alpha_1 + 3\alpha_2$
$\Rightarrow \alpha_1 = 1, \alpha_2 = 2$
$\Rightarrow a_n = 3^n + 2n \cdot 3^n$

5. (15 points) Solve the following recurrence relation: $a_n = 3a_{n-1} + 2$, $a_1 = 1$. Express your answer as a closed for solution which is a function of n. (Try to prove your solution by induction.)
Answer: $a_n = 2 \cdot 3^{n-1} - 1$

$$
\begin{aligned}
a_n &= 3a_{n-1} + 2 \\
&= 3(3a_{n-2} + 2) + 2 && = 3^2 a_{n-2} + 2(1+3) \\
&= 3^2(3a_{n-3} + 2) + 2(1+3) && = 3^3 a_{n-3} + 2(1+3+3^2) \\
&\vdots \\
&= 3^{n-1} a_{n-(n-1)} + 2(1+3+3^2+...+3^{n-2}) \\
&= 3^{n-1} a_1 + 2\tfrac{3^{n-1}-1}{3-1} \\
&= 3^{n-1} + (3^{n-1} - 1) \\
&= 2 \cdot 3^{n-1} - 1
\end{aligned}
$$

Proof by induction: Let P(n) be "$a_n = 2 \cdot 3^{n-1} - 1$", where $n = 1, 2, \cdots$

- Basis step: for n = 1, $a_1 = 2 \cdot 3^{1-1} - 1 = 1 \Rightarrow$ P(1) is true.
- Inductive step: Assume P(n) is true, i.e. $a_n = 2 \cdot 3^{n-1} - 1$, then
  $a_{n+1} = 3a_n + 2$
  $= 3 \cdot (2 \cdot 3^{n-1} - 1) + 2$
  $= 3 \cdot 2 \cdot 3^{n-1} - 3 + 2 = 2 \cdot 3^n - 1$

The last line shows that P(n+1) is true. This completes the inductive step and completes the proof.

6. (10 points) How many sequences of five letters are there in which exactly two are vowels. "Y" does not count as a vowel and we are not distinguishing between lower and upper case.
Answer: 21 consonants and 5 vowels in English. The sequence is constitute of 2 vowels and 3 consonants.
number of possible positions for the vowel: $C(5,2)$
so the number of sequences with exact two vowels: $C(5,2) \cdot 5^2 \cdot 21^3$

7. (15 points) I have a 100 comic books which I am dispensing at random to ten kids. That is, each comic book is given to one of the ten kids chosen at random. Note that kids are distinct and the comic books are distinct.

   (a) (5 points) What is the probability that a specific kid (say George) gets no comic book at all?
   Answer: We can think the distribution of 100 books as 100 trials. In each trial I give one book to one kid. And these 100 trials are independent.
   - For each trial, the probability that George gets the book (success) is: $\frac{1}{10}$.
   - For each trial, the probability that George does not get the book (fail) is: $1 - \frac{1}{10} = \frac{9}{10}$.

- After 100 trial, the probability that George still get nothing is: $C(100,100) \cdot (\frac{1}{10})^0 \cdot (1-\frac{1}{10})^{100} = (\frac{9}{10})^{100}$

(b) (10 points) What is the probability that there is some kid who gets no comic book?
Answer:

(c) (15 points) What is the expected number of comic books that George gets? (Full credit for a numeric answer).
Answer: 10 books.
If there is only one book, the expected number of books that George can get is: $1 \cdot \frac{1}{10} + 0 \cdot (1-\frac{1}{10}) = \frac{1}{10}$.
So if there is 100 books, the expected number of books that George can get is: $100 \cdot \frac{1}{10} = 10$.

8. *(35 points)*

(a) A pair of dice is rolled once. What is the probability that the total number of points on the dice is seven?
Answer: there are 6 elements : (1,6)(2,5)(3,4)(4,3)(5,2)(6,1) $p = 6/36 = 1/6$

(b) What is the expected number of times we get a total of 7 in one roll?
Answer: $E = 0 \cdot (1-\frac{1}{6}) + 1 \cdot \frac{1}{6} = \frac{1}{6}$

(c) Now suppose that a pair of dice is rolled twice. You can assume that the outcomes of the two rolls are independent.

(d) What is the probability that exactly one of the two rolls comes up seven?
Answer: $P = 2 \cdot \frac{1}{6} \cdot (1-\frac{1}{6}) = \frac{5}{18}$

(e) What is the probability that at least one of the two rolls comes up seven?
Answer: $P = 1-(1-\frac{1}{6})(1-\frac{1}{6}) = \frac{11}{36}$

(f) What is the probability that both rolls come up seven, conditioned on the fact that at least one roll comes up seven?
Answer: The probability of both rolls coming up 7 and and least one roll come up 7 is: $(1/6)^2 = 1/36$.
The probability of at least one roll come up 7 is the result of (e): $\frac{11}{36}$.
Therefore the answer is the ratio $\frac{\frac{1}{36}}{\frac{11}{36}} = \frac{1}{11}$.

(g) What is the expected number of times that we get a total of seven in two independent rolls?
Answer: $2 \cdot result\ of\ (b) = 2 \cdot \frac{1}{6} = \frac{1}{3}$

9. *(10 points)* The odds in playing a slot machine are as follows: the probability that the player loses his quarter is 9/10. The probability that four quarters comes back is 9/100. The probability of hitting the jackpot (forty quarters) is 1/100. You can

assume that the outcome of one game is independent of the outcome of any other game. Suppose that a player sits down and plays the game 100 times. Let X be the random variable which denotes the number of times the player hits the jackpot. Let Y denote the number of times the player loses his quarter. Are X and Y independent? Prove your answer.

Answer: NO, X and Y are NOT independent, i.e. they are dependent.

- The probability that there are x times the player hits the jackpot in the 100 times is:
  $P(X = x) = C(100, x)(\frac{1}{100})^x(1 - \frac{1}{100})^{100-x} = C(100, x)(\frac{1}{100})^x(\frac{99}{100})^{100-x}$
- The probability that there are y times the player loses his quarter in the 100 times is:
  $P(Y = y) = C(100, y)(\frac{9}{10})^y(1 - \frac{9}{10})^{100-y} = C(100, y)(\frac{9}{10})^y(\frac{1}{10})^{100-y}$
- The probability that there are x times the player hits the jackpot AND y times he loses his quarter in the 100 times, hence there are $100 - x - y$ times the player win four quarters, is:
  $P((X = x) \cap (Y = y)) = C(100, x)C(100 - x, y)(\frac{1}{100})^x(\frac{9}{10})^y(\frac{9}{100})^{100-x-y}$

We can get $P(X = x) \cdot P(Y = y) \neq P((X = x) \cap (Y = y))$ after some algebraic simplification, so X and Y are NOT independent, i.e they are dependent.

10. (30 points)

    (a) Sam has just joined Chocoholics Anonymous in an attempt to quit eating chocolate. He normally eats n M&Ms a day. He decides to go on a program in which every week, he cuts down his daily M&M intake by one half. (That is, if he eats n M&Ms a day for one week, then he eats $n/2$ M&Ms a day for the next week, etc.) Including the first week when he is eating his usual n M&Ms a day, how many weeks will it take him until he is down to at most one M&M a day?

    Solution: Let a_k be the number of M&Ms Sam eats a day in the kth week, then $a_k = a_{k-1}/2$, with $a_1 = n \Rightarrow a_k = n \cdot \frac{1}{2}^{k-1}$. Suppose it will take Sam m weeks until he is down to at most one M&M a day, then
    $a_m = 1 \Rightarrow n \cdot \frac{1}{2}^{m-1} = 1$
    $\Rightarrow 2^{m-1} = n$
    $\Rightarrow m = \lceil log_2^n \rceil + 1$

    (b) Assume now that n is a power of two. How many M&Ms will he have eaten during the whole reduction program? You should include the first week in which he eats his usual n M&Ms a day. You can assume he stops eating M&Ms after the week where he eats only one M&M a day.

    Solution: $\sum_{i=1}^{i=m}(7 \cdot a_i)$
    $= \sum_{i=1}^{i=m}(7 \cdot n \cdot \frac{1}{2}^{i-1})$
    $= 7n \cdot \sum_{i=1}^{i=m}(\frac{1}{2}^{i-1})$
    $= 7n \cdot \frac{1-(\frac{1}{2})^m}{1-\frac{1}{2}}$
    $= 7n \cdot 2(1 - \frac{1}{2n}$ Since $2^{m-1} = n$ from part (a)

$= 7(2n - 1)$

(c) Now, suppose, he opts for a different program in which each day he eats three fewer M&Ms than the day before. You can assume that n is a multiple of three and that the program ends when he is down to 0 M&Ms a day. How many M&Ms does he eat over the course of this program? You should include the first day in which he eats his usual n M&Ms.

Solution: Let a_k be the number of M&Ms Sam eats a day in the kth week, then $a_k = a_{k-1} - 3$, with $a_1 = n \Rightarrow a_k = n + 3 - 3k$. Suppose it will take Sam m weeks until he is down to 0 M&M a day, then

$a_m = 0 \Rightarrow n + 3 - 3m = 0$

$\Rightarrow 3m = n + 3$

$\Rightarrow m = \lceil \frac{n}{3} \rceil + 1$

$\sum_{i=1}^{i=m} (7 \cdot a_i)$

$= \sum_{i=1}^{i=m} (7 \cdot (n + 3 - 3i)$

$= 7(m(n + 3) - 3 \sum_{i=1}^{i=m} i)$

$= 7(m(n + 3) - 3\frac{m(m+1)}{2}$

$= \frac{7n(n+3)}{6}$ Since $3m = n + 3$