Bounded inference non-iteratively; Mini-bucket elimination

COMPSCI 276, Spring 2013

Set 10: Rina Dechter

(Reading: Primary: Class Notes (10)
Secondary: , Darwiche chapters 14)

Agenda

- Mini-bucket elimination
- Mini-clustering
- Iterative Belief propagation
- Iterative-join-graph propagation

Probabilistic Inference Tasks

- Belief updating:

$$\mathbf{BEL(X_i) = P(X_i = x_i \mid evidence)}$$

- Finding most probable explanation (MPE)

$$\bar{x}^* = \arg\max_{\bar{x}} P(\bar{x}, e)$$

- Finding maximum a-posteriory hypothesis

$$(a_1^*, \ldots, a_k^*) = \arg\max_{\bar{a}} \sum_{X/A} P(\bar{x}, e)$$

$A \subseteq X$: **hypothesis variables**

- Finding maximum-expected-utility (MEU) decision

$$(d_1^*, \ldots, d_k^*) = \arg\max_{\bar{d}} \sum_{X/D} P(\bar{x}, e) U(\bar{x})$$

$D \subseteq X$: **decision variables**

$U(\bar{x})$: **utility function**

Queries

- **Probability of evidence (or partition function)**

$$P(e) = \sum_{X - \mathrm{var}(e)} \prod_{i=1}^{n} P(x_i \mid pa_i)|_e \qquad Z = \sum_X \prod_i \psi_i(C_i)$$

- **Posterior marginal (beliefs):**

$$P(x_i \mid e) = \frac{P(x_i, e)}{P(e)} = \frac{\sum_{X - \mathrm{var}(e) - X_i} \prod_{j=1}^{n} P(x_j \mid pa_j)|_e}{\sum_{X - \mathrm{var}(e)} \prod_{j=1}^{n} P(x_j \mid pa_j)|_e}$$

- **Most Probable Explanation**

$$\bar{x}^* = \arg\max_{\bar{x}} P(\bar{x}, e)$$

Finding $MPE = \max_{\bar{x}} P(\bar{x})$

Algorithm *elim-mpe* (Dechter 1996)

$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)P(d|a,b)P(e|b,c)$$

$\max_b \prod$ ← Elimination operator

bucket B: $\underbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$

bucket C: $P(c|a)$ The image cannot be displayed. Your computer may not have enough memory to open the image, or the image may have been corrupted. Restart your computer, and then open the file again. If the red x still appears, you may have to delete the image and then insert it again.

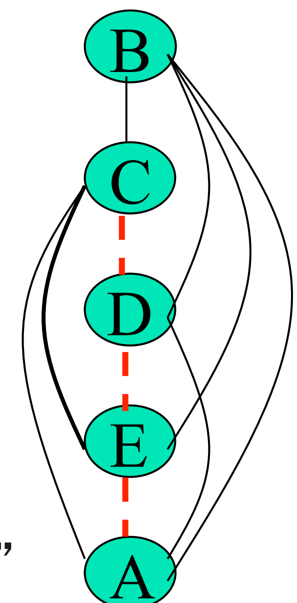
bucket D: $h^C(a,d,e)$

bucket E: $e=0$ $\quad h^D(a,e)$

bucket A: $P(a)$ $\quad h^E(a)$

MPE

W*=4
"induced width"
(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_c P(c \mid a') \times h^B(a', d', c, e')$

3. $d' = \arg\max_d h^C(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_a P(a) \cdot h^E(a)$

B: P(b|a) P(d|b,a) P(e|b,c)

C: P(c|a) $h^B(a,d,c,e)$

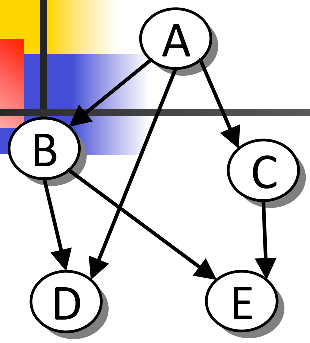
D: $h^C(a,d,e)$

E: e=0 $h^D(a,e)$

A: P(a) $h^E(a)$

Return (a', b', c', d', e')

Bucket Elimination



Query: $P(a \mid e=0) \propto P(a, e=0)$ Elimination Order: d,e,b,c

$$P(a, e=0) = \sum_{c,b,e=0,d} P(a)P(b \mid a)P(c \mid a)P(d \mid a,b)P(e \mid b,c)$$

$$= P(a)\sum_{c} P(c \mid a)\sum_{b} P(b \mid a)\sum_{e=0} P(e \mid b,c)\sum_{d} P(d \mid a,b)$$

Original Functions

D: $P(d \mid a,b)$

E: $P(e \mid b,c)$

B: $P(b \mid a)$

C: $P(c \mid a)$

A: $P(a)$

Messages

$$\boxed{f_D(a,b)} = \sum_{d} P(d \mid a,b)$$

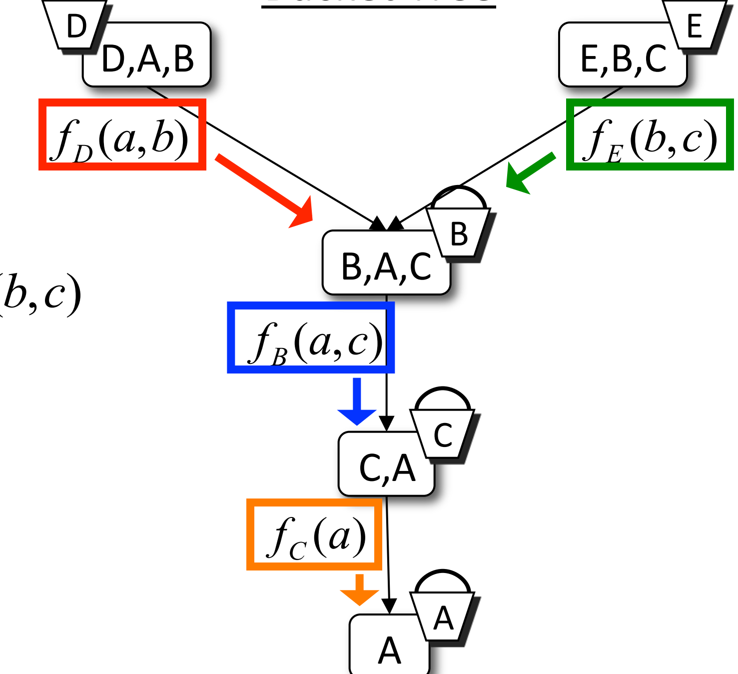
$$\boxed{f_E(b,c)} = P(e=0 \mid b,c)$$

$$\boxed{f_B(a,c)} = \sum_{b} P(b \mid a) f_D(a,b) f_E(b,c)$$

$$\boxed{f_C(a)} = \sum_{c} P(c \mid a) f_B(a,c)$$

$$P(a, e=0) = p(A) f_C(a)$$

Bucket Tree



Time and space exp(w*)

Approximate Inference

- Metrics of evaluation
- **Absolute error**: given e>0 and a query p= P(x|e), an estimate r has absolute error e iff |p-r|<e
- **Relative error**: the ratio r/p in [1-e,1+e].
- Dagum and Luby 1993: approximation up to a relative error is NP-hard.
- Absolute error is also NP-hard if error is less than .5

Mini-buckets: “local inference”

- Computation in a bucket is time and space exponential in the number of variables involved

- Therefore, partition functions in a bucket into “mini-buckets” on smaller number of variables

Mini-bucket approximation: MPE task

Split a bucket into mini-buckets =>bound complexity

$$\mathbf{bucket}(X) = \{ h_1, \ldots, h_r, h_{r+1}, \ldots, h_n \}$$

$$h^X = \max_X \prod_{i=1}^{n} h_i$$

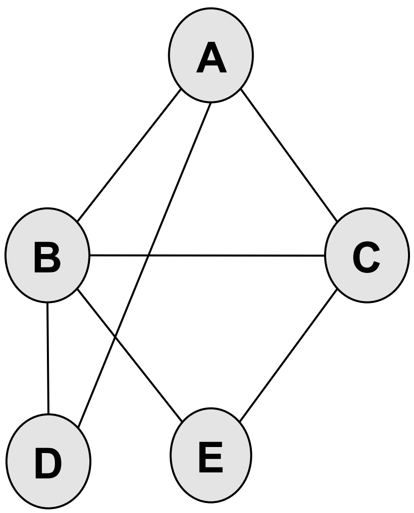
$$\{ h_1, \ldots, h_r \} \qquad \{ h_{r+1}, \ldots, h_n \}$$

$$g^X = \left(\max_X \prod_{i=1}^{r} h_i \right) \cdot \left(\max_X \prod_{i=r+1}^{n} h_i \right)$$

$$\boxed{h^X \le g^X}$$

Exponential complexity decrease: $O(e^n) \rightarrow O(e^r) + O(e^{n-r})$

Mini-Bucket Elimination



Mini-buckets

$\min_B \Sigma$ $\quad$ $\min_B \Sigma$

bucket B: $\underbrace{F(a,b)\,F(b,c)}$ $\quad$ $\underbrace{F(b,d)\,F(b,e)}$

bucket C: $\underbrace{\boldsymbol{h^B(a,c)}\;\; F(c,e)\;\; F(a,c)}$

bucket D: $\underbrace{F(a,d)\;\; \boldsymbol{h^B(d,e)}}$

bucket E: $\underbrace{e = 0\;\; \boldsymbol{h^C(e,a)}\;\; \boldsymbol{h^D(e,a)}}$

bucket A: $\underbrace{\boldsymbol{h^E(a)}}$

L = lower bound

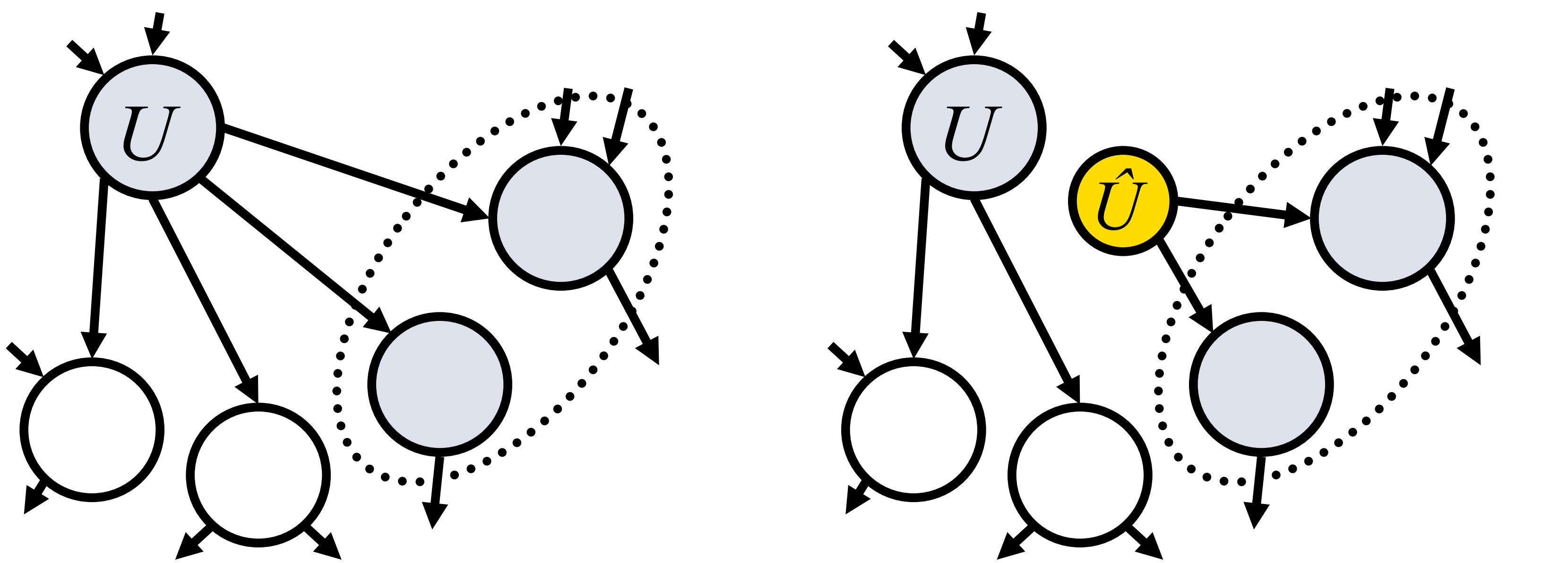
We can generate a solution s going forward as before U= F(s)

Semantics of Mini-Bucket: Splitting a Node

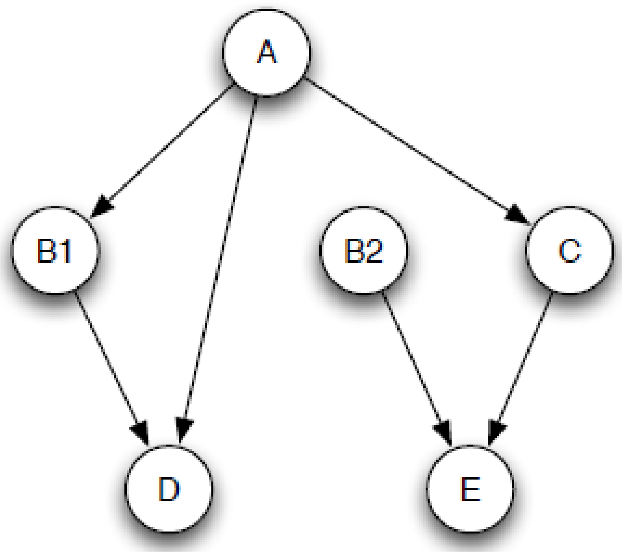
Variables in different buckets are renamed and duplicated
(Kask *et. al.*, 2001), (Geffner *et. al.*, 2007), (Choi, Chavira, Darwiche , 2007)

Before Splitting:
Network N

After Splitting:
Network N'



Relaxed network example



(a)

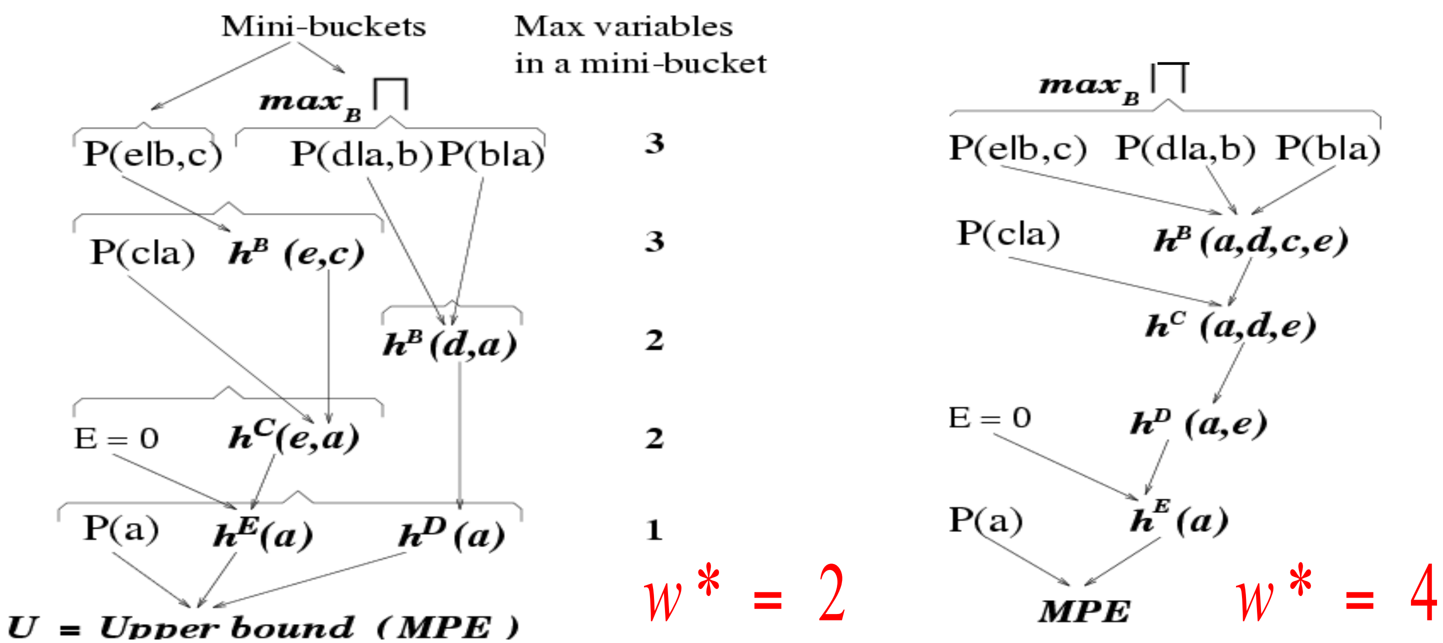
B1: $P(b1|a), P(d|b1,a)$
B2: $P(e|b2,c)$
C: $P(c|a)$
D:
E: $E=e$
A: $P(a)$

(b)

MBE-mpe(i)

- Input: i – max number of variables allowed in a mini-bucket
- Output: [lower bound (Probability of a sub-optimal solution), upper bound]

Example: approx-mpe(3) versus elim-mpe



(i,m) patitionings

Definition 7.1.1 ((i,m)-partitioning) *Let H be a collection of functions $h_1, ..., h_t$ defined on scopes $S_1, ..., S_t$, respectively. We say that a function f is* subsumed *by a function h if any argument of f is also an argument of h. A partitioning of $h_1, ..., h_t$ is* canonical *if any function f subsumed by another function is placed into the bucket of one of those subsuming functions. A partitioning Q into mini-buckets is an (i,m)-partitioning if and only if (1) it is canonical, (2) at most m non-subsumed functions are included in each mini-bucket, (3) the total number of variables in a mini-bucket does not exceed i, and (4) the partitioning is* refinement-maximal, *namely, there is no other (i,m)-partitioning that it refines.*

MBE(i,m), MBE(i)

- Input: Belief network (P1,…Pn)
- Output: upper and lower bounds
- Initialize: (put functions in buckets)
- Process each bucket from p=n to 1
  - Create (i,m)-mini-buckets partitions
  - Process each mini-bucket
- (For mpe): assign values in ordering d
- Return: mpe-tuple, upper and lower bounds

Algorithm mbe-mpe(i,m)

Input: A belief network $BN = (G, P)$, an ordering o, evidence $\bar{e}$.

Output: An upper bound U and a lower bound L on the $MPE = \max_{\bar{x}} P(\bar{x}, \bar{e})$, and a suboptimal solution $\bar{x}^a$ that provides $L = P(\bar{x}^a)$.

1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into buckets $bucket_1, \ldots, bucket_n$, where $bucket_p$ contains all CPTs $h_1, h_2, ..., h_t$ whose highest-index variable is X_p.
2. **Backward:** for $p = n$ to 2 do
   - If X_p is observed ($X_p = a$), assign $X_p = a$ in each h_j and put the result in its highest-variable bucket (put constants in $bucket_1$).
   - Else for $h_1, h_2, ..., h_t$ in $bucket_p$ do
     Generate an (i,m)-mini-bucket-partitioning, $Q' = \{Q_1, ..., Q_r\}$.
     for each $Q_l \in Q'$ containing $h_{l_1}, ...h_{l_t}$, do
        compute $h^l = max_{X_p} \Pi_{j=1}^{t} h_{l_j}$ and place it in the bucket of the highest-index variable in $U_l \leftarrow \bigcup_{j=1}^{t} S_{l_j} - \{X_p\}$, where S_{l_j} is the scope of h_{l_j}
        (put constants in $bucket_1$).
3. **Forward:** for $p = 1$ to n, given $x_1^a, ..., x_{p-1}^a$, do assign a value x_p^a to X_p that maximizes the product of all functions in $bucket_p$.
4. **Return** the assignment $\bar{x}^a = (x_1^a, ..., x_n^a)$, a lower bound $L = P(\bar{x}^a)$, and an upper bound $U = max_{x_1} \prod_{h_j \in bucket_1} h^j$ on the $MPE = \max_{\bar{x}} P(\bar{x}, \bar{e})$.

Theorem 7.1.3 (mbe-mpe properties) *Algorithm* mbe-mpe(i,m) *computes an upper bound on the MPE. Its time and space complexity is $O(n \cdot exp(i))$ where $i \leq n$.*

Partitioning refinements

Clearly, as the mini-buckets get smaller, both complexity and accuracy decrease.

Definition 7.1.4 *Given two partitionings Q' and Q'' over the same set of elements, Q' is a refinement of Q'' if and only if for every set $A \in Q'$ there exists a set $B \in Q''$ such that $A \subseteq B$.*

It is easy to see that:

Proposition 7.1.5 *If Q'' is a refinement of Q' in bucket$_p$, then $h^p \leq g^p_{Q'} \leq g^p_{Q''}$.*

Remember that *mbe-mpe* computes the bounds on $MPE = \max_{\bar{x}} P(\bar{x}, \bar{e})$, rather than on $M = \max_{\bar{x}} P(\bar{x}|\bar{e}) = MPE/P(\bar{e})$. Thus

$$\frac{L}{P(\bar{e})} \leq M \leq \frac{U}{P(\bar{e})}$$

Properties of MBE-mpe(i)

- **Complexity:** *O(exp(i))* time and *O(exp(i))* space.
- **Accuracy:** determined by upper/lower (U/L) bound.
- As ***i*** increases, both accuracy and complexity increase.
- Possible use of mini-bucket approximations:
  - As anytime algorithms
  - As heuristics in best-first search

Anytime Approximation

Algorithm anytime-mpe(ϵ)
Input: Initial values of i and m, i_0 and m_0; increments i_{step} and m_{step}, and desired approximation error ϵ.
Output: U and L

1. **Initialize:** $i = i_0, m = m_0$.
2. **do**
3. run *mbe-mpe(i,m)*
4. $U \leftarrow$ upper bound of *mbe-mpe(i,m)*
5. $L \leftarrow$ lower bound of *mbe-mpe(i,m)*
6. Retain best bounds U, L, and best solution found so far
7. **if** $1 \leq U/L \leq 1 + \epsilon$, return solution
8. **else** increase i and m: $i \leftarrow i + i_{step}$ and $m \leftarrow m + m_{step}$
9. **while** computational resources are available
10. **Return** the largest L and the smallest U found so far.

MBE for Belief Updating and for probability of evidence

- Idea mini-bucket is the same:

$$\sum_X f(x) \bullet g(x) \le \sum_X f(x) \bullet \sum_X g(x)$$

$$\sum_X f(x) \bullet g(x) \le \sum_X f(x) \bullet \max_X g(X)$$

- So we can apply a sum in each mini-bucket, or better, one sum and the rest max, or min (for lower-bound)
- MBE-bel-max(i,m), MBE-bel-min(i,m) generating upper and lower-bound on beliefs approximates BE-bel
- MBE-map(i,m): max buckets will be maximized, sum buckets will be sum-max. Approximates BE-map.

Normalization

- mbe-bel computes upper/lower bound on the joint marginal distributions.

Alternatively, let U_i and L_i be the upper bound and lower bounding functions on $P(X_1 = x_i, \bar{e})$ obtained by *mbe-bel-max* and *mbe-bel-min*, respectively. Then,

$$\frac{L_i}{P(\bar{e})} \leq P(x_i|\bar{e}) \leq \frac{U_i}{P(\bar{e})}$$

We sometime use normalization of the approximation, but then no guarantee.
The probable is that we have to approximate also the partition function.

Algorithm mbe-bel-max(i,m)

Algorithm mbe-bel-max(i,m)
Input: A belief network $BN = (G, P)$, an ordering o, and evidence $\bar{e}$.
Output: an upper bound on $P(x_1, \bar{e})$ and an upper bound on $P(e)$.
1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into buckets $bucket_1, \ldots, bucket_n$, where $bucket_k$ contains all CPTs $h_1, h_2, ..., h_t$ whose highest-index variable is X_k.
2. **Backward:** for $k = n$ to 2 do
   - **If** X_p is observed ($X_k = a$), assign $X_k \leftarrow a$ in each h_j and put the result in the highest-variable bucket of its scope (put constants in $bucket_1$).
   - **Else** for $h_1, h_2, ..., h_t$ in $bucket_k$ do
     Generate an (i, m)-mini-bucket-partitioning, $Q' = \{Q_1, ..., Q_r\}$.
     For each $Q_l \in Q'$, containing $h_{l_1}, ...h_{l_t}$, do
       If $l = 1$ compute $h^l = \sum_{X_k} \Pi_{j=1}^t h_{1_j}$
       Else compute $h^l = max_{X_k} \Pi_{j=1}^t h_{l_j}$
       Add h^l to the bucket of the highest-index variable in $U_l \leftarrow \bigcup_{j=1}^t S_{l_j} - \{X_k\}$,
       (put constant functions in $bucket_1$).
3. **Return** $P'(\bar{x}_1, e) < --$ the product of functions in the bucket of X_1, which is an upper bound on $P(x_1, \bar{e})$.
$P'(e) < -- \sum_{x_1} P'(\bar{x}_1, e)$, which is an upper bound on probability of evidence.

Empirical Evaluation
(Dechter and Rish, 1997; Rish thesis, 1999)

- Randomly generated networks
  - Uniform random probabilities
  - Random noisy-OR
- CPCS networks
- Probabilistic decoding

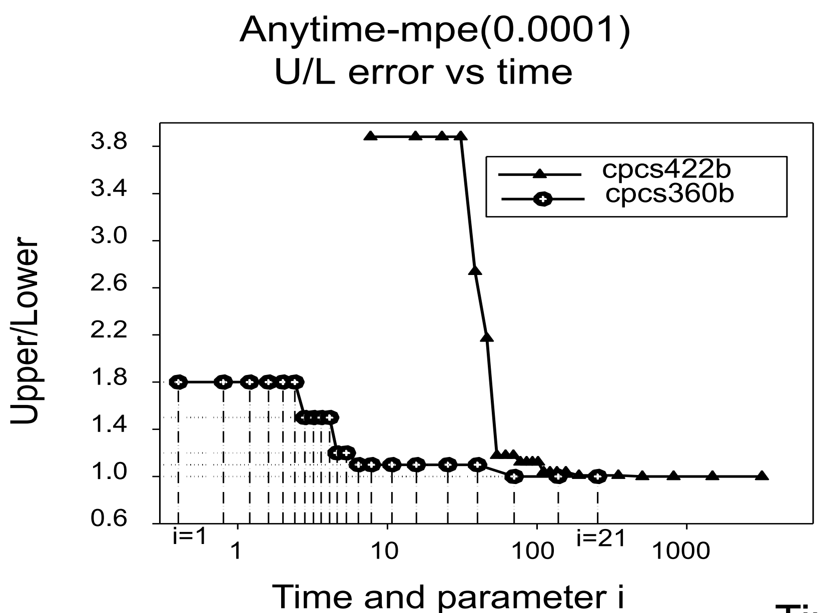
Comparing MBE-mpe and anytime-mpe
versus BE-mpe

Methodology for Empirical Evaluation (for mpe)

- U/L –accuracy
- Better (U/mpe) or mpe/L
- Benchmarks: Random networks
  - Given n,e,v generate a random DAG
  - For xi and parents generate table from uniform [0,1], or noisy-or
- Create k instances. For each, generate random evidence, likely evidence
- Measure averages

CPCS networks – medical diagnosis (noisy-OR model)

Test case: no evidence

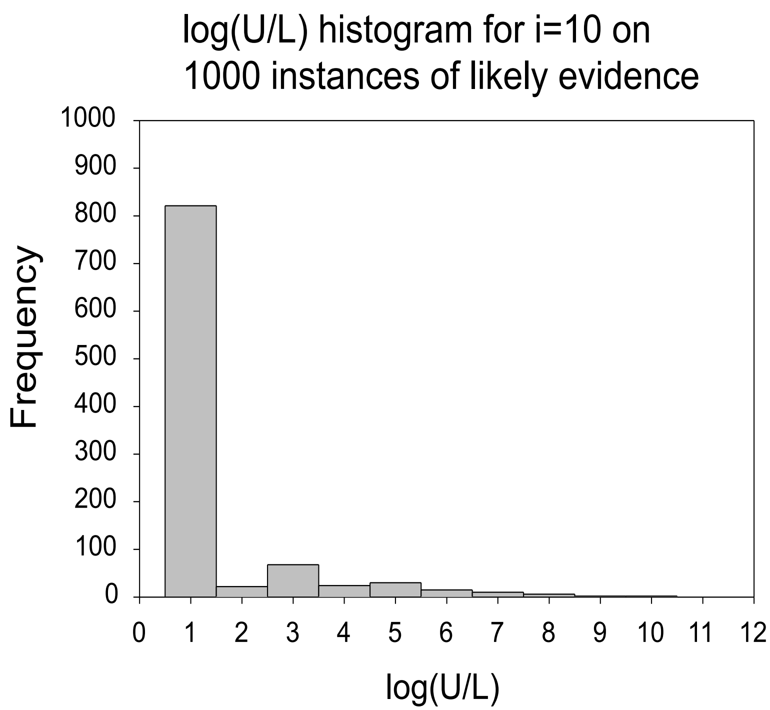


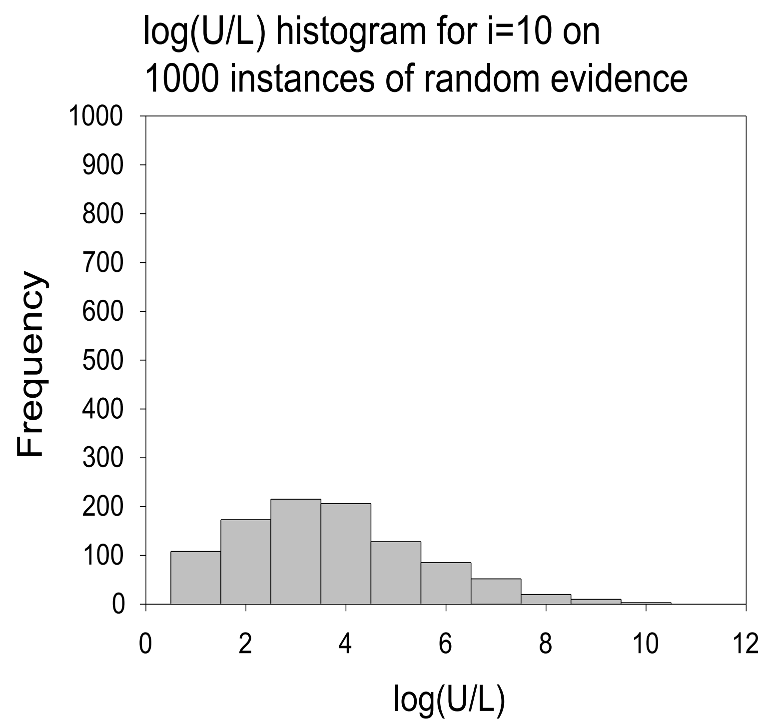
Time (sec)

Algorithm	cpcs360	cpcs422
elim-mpe	115.8	1697.6
anytime-mpe(ε), $\varepsilon = 10^{-4}$	70.3	505.2
anytime-mpe(ε), $\varepsilon = 10^{-1}$	70.3	110.5

The effect of evidence

More likely evidence=>higher MPE => higher accuracy (why?)





Likely evidence versus random (unlikely) evidence

MBE-map

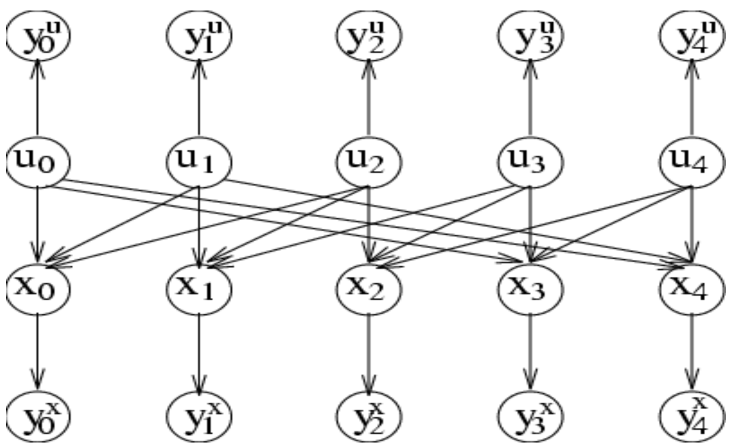
Process max buckets
With max mini-buckets
And sum buckets with sum
Mini-bucket and max
mini-buckets

Algorithm mbe-map(i,m)
Input: A belief network $BN = (G, P)$, a subset of variables $A = \{A_1, ..., A_k\}$, an ordering of the variables, o, in which the A's appear first, and evidence $\bar{e}$.
Output: An upper bound U on the MAP and a suboptimal solution $A = \bar{a}_k^a$.
1. **Initialize:** Partition $P = \{P_1, ..., P_n\}$ into buckets $bucket_1, \ldots, bucket_n$ where $bucket_P$ contains all CPTs, $h_1, ..., h_t$ whose highest index variable is X_p.
2. **Backward:** for $p = n$ to 1 do
   - **If** X_p is observed ($X_p = a$), assign $X_p = a$ in each h_i and put the result in its highest-variable bucket (put constants in $bucket_1$).
   - **Else** for $h_1, h_2, ..., h_j$ in $bucket_p$ do
     Generate an (i, m)-partitioning, Q' of the matrices h_i into mini-buckets $Q_1, ..., Q_r$.
     - **If** $X_P \notin A$ /* not a hypothesis variable */
       for each $Q_l \in Q'$, containing $h_{l_1}, ...h_{l_t}$, do
       If $l = 1$, compute $h^l = \sum_{X_p} \Pi_{i=1}^t h_{1_i}$
       Else compute $h^l = max_{X_p} \Pi_{i=1}^t h_{l_i}$
       Add h^l to the bucket of the highest-index variable in $U_l \leftarrow \bigcup_{i=1}^t S_{l_i} - \{X_p\}$, (put constants in $bucket_1$).
     - **Else** $(X_p \in A)$ /* a hypothesis variable */
       for each $Q_l \in Q'$ containing $h_{l_1}, ...h_{l_t}$ compute $h^l = max_{X_p} \Pi_{i=1}^t h_{l_i}$ and place it in the bucket of the highest-index variable in $U_l \leftarrow \bigcup_{i=1}^t S_{l_i} - \{X_p\}$, (put constants in $bucket_1$).
3. **Forward:** for $p = 1$ to k, given $A_1 = a_1^a, ..., A_{p-1} = a_{p-1}^a$, assign a value a_p^a to A_p that maximizes the product of all functions in $bucket_p$.
4. **Return** An upper bound $U = max_{a_1} \prod_{h_i \in bucket_1} h_i$ on MAP, computed in the first bucket. and the assignment $\bar{a}_k^a = (a_1^a, ..., a_k^a)$.

Figure 7.6: Algorithm *mbe-map(i,m)*.

Probabilistic decoding

Error-correcting linear block code



State-of-the-art:
approximate algorithm – iterative belief propagation (IBP)
(Pearl's poly-tree algorithm applied to loopy networks)

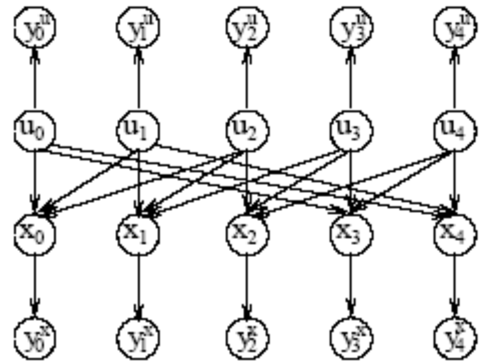


Figure 7.7: Belief network for a linear block code.

Example 7.3.1 We will next demonstrate the mini-bucket approximation for MAP on an example of *probabilistic decoding* (see Chapter 2) Consider a belief network which describes the decoding of a *linear block code*, shown in Figure 7.7. In this network, U_i are *information bits* and X_j are *code bits*, which are functionally dependent on U_i. The vector (U, X), called the channel input, is transmitted through a noisy channel which adds Gaussian noise and results in the channel output vector $Y = (Y^u, Y^x)$. The decoding task is to assess the most likely values for the U's given the observed values $Y = (\bar{y}^u, \bar{y}^x)$, which is the MAP task where U is the set of hypothesis variables, and $Y = (\bar{y}^u, \bar{y}^x)$ is the evidence. After processing the observed buckets we get the following bucket configuration (lower case y's are observed values):

$bucket(X_0) = P(y_0^x|X_0), P(X_0|U_0, U_1, U_2),$
$bucket(X_1) = P(y_1^x|X_1), P(X_1|U_1, U_2, U_3),$
$bucket(X_2) = P(y_2^x|X_2), P(X_2|U_2, U_3, U_4),$
$bucket(X_3) = P(y_3^x|X_3), P(X_3|U_3, U_4, U_0),$
$bucket(X_4) = P(y_4^x|X_4), P(X_4|U_4, U_0, U_1),$
$bucket(U_0) = P(U_0), P(y_0^u|U_0),$
$bucket(U_1) = P(U_1), P(y_1^u|U_1),$
$bucket(U_2) = P(U_2), P(y_2^u|U_2),$
$bucket(U_3) = P(U_3), P(y_3^u|U_3),$
$bucket(U_4) = P(U_4), P(y_4^u|U_4).$

Initial partitioning

Processing by *mbe-map(4,1)* of the first top five buckets by summation and the rest by maximization, results in the following mini-bucket partitionings and function generation:

$bucket(X_0) = \{P(y_0^x|X_0), P(X_0|U_0, U_1, U_2)\},$
$bucket(X_1) = \{P(y_1^x|X_1), P(X_1|U_1, U_2, U_3)\},$
$bucket(X_2) = \{P(y_2^x|X_2), P(X_2|U_2, U_3, U_4)\},$
$bucket(X_3) = \{P(y_3^x|X_3), P(X_3|U_3, U_4, U_0)\},$
$bucket(X_4) = \{P(y_4^x|X_4), P(X_4|U_4, U_0, U_1)\},$
$bucket(U_0) = \boxed{\{P(U_0), P(y_0^u|U_0), h^{X_0}(U_0, U_1, U_2)\}}, \boxed{\{h^{X_3}(U_3, U_4, U_0)\}}, \boxed{\{h^{X_4}(U_4, U_0, U_1)\}},$
$bucket(U_1) = \{P(U_1), P(y_1^u|U_1), h^{X_1}(U_1, U_2, U_3), h^{U_0}(U_1, U_2)\}, \{h^{U_0}(U_4, U_1)\},$
$bucket(U_2) = \{P(U_2), P(y_2^u|U_2), h^{X_2}(U_2, U_3, U_4), h^{U_1}(U_2, U_3)\},$
$bucket(U_3) = \{P(U_3), P(y_3^u|U_3), h^{U_0}(U_3, U_4), h^{U_1}(U_3, U_4), h^{U_2}(U_3, U_4)\},$
$bucket(U_4) = \{P(U_4), P(y_4^u|U_4), h^{U_1}(U_4), h^{U_3}(U_4)\}.$

The first five buckets are not partitioned at all and are processed as full buckets, since in this case a full bucket is a (4,1)-partitioning. This processing generates five new functions, three are placed in bucket U_0, one in bucket U_1 and one in bucket U_2. Then bucket U_0 is partitioned into three mini-buckets processed by maximization, creating two functions placed in bucket U_1 and one function placed in bucket U_3. Bucket U_1 is partitioned into two mini-buckets, generating functions placed in bucket U_2 and bucket U_3. Subsequent buckets are processed as full buckets. Note that the scope of recorded functions is bounded by 3.

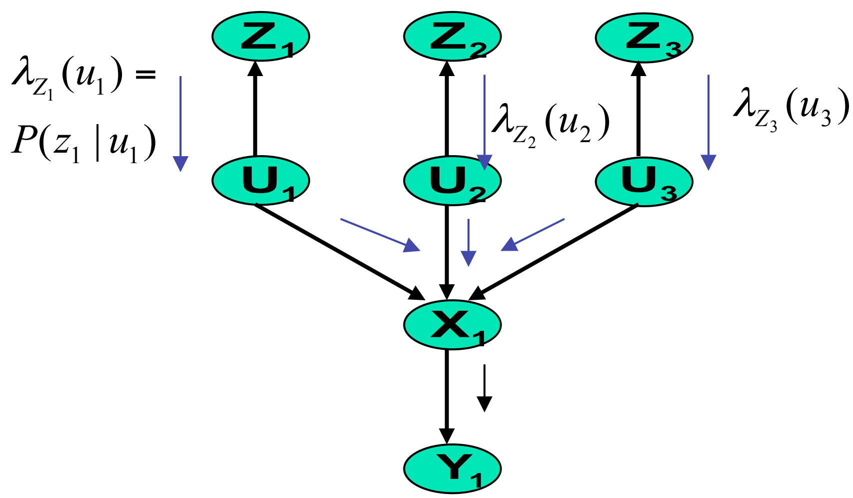
In the bucket of U_4 we get an upper bound U satisfying $U \geq MAP = P(U, \bar{y}^u, \bar{y}^x)$ where $\bar{y}^u$ and ,$\bar{y}^x$ are the observed outputs for the U's and the X's bits transmitted. In order to bound $P(U|\bar{e})$, where $\bar{e} = (\bar{y}^u, \bar{y}^x)$, we need $P(\bar{e})$ which is not available. Yet, again, in most cases we are interested in the ratio $P(U = \bar{u}_1|\bar{e})/P(U = \bar{u}_2|\bar{e})$ for competing hypotheses $U = \bar{u}_1$ and $U = \bar{u}_2$ rather than in the absolute values. Since $P(U|\bar{e}) = P(U, \bar{e})/P(\bar{e})$ and the probability of the evidence is just a constant factor independent of U, the ratio is equal to $P(U_1, \bar{e})/P(U_2, \bar{e})$. □

Complexity and tractability of MBE(i,m)

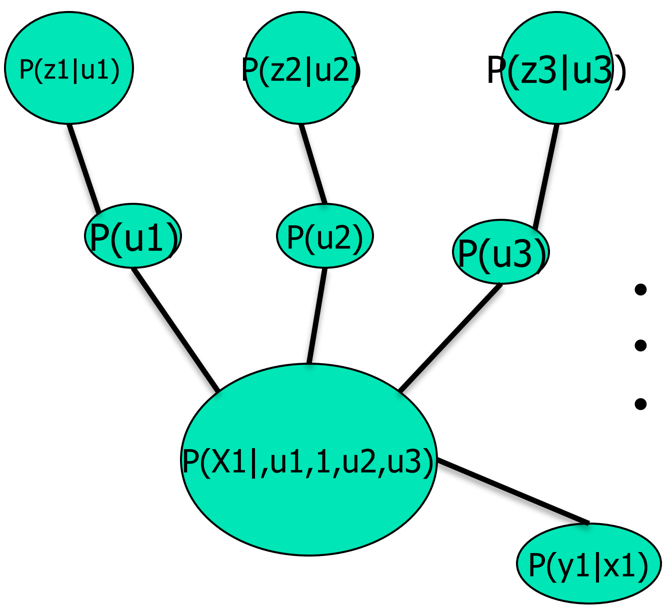
Theorem 7.6.1 *Algorithm* mbe(i,m) *takes* $O(r \cdot exp(i))$ *time and space, where* r *is the number of input functions*[2], *and where* $|F|$ *is the maximum scope of any input function,* $|F| \leq i \leq n$. *For* $m = 1$, *the algorithm is time and space* $O(r \cdot exp(|F|))$.

Belief propagation is easy on polytree: Pearl's Belief Propagation

A polytree: a tree with
Larger families



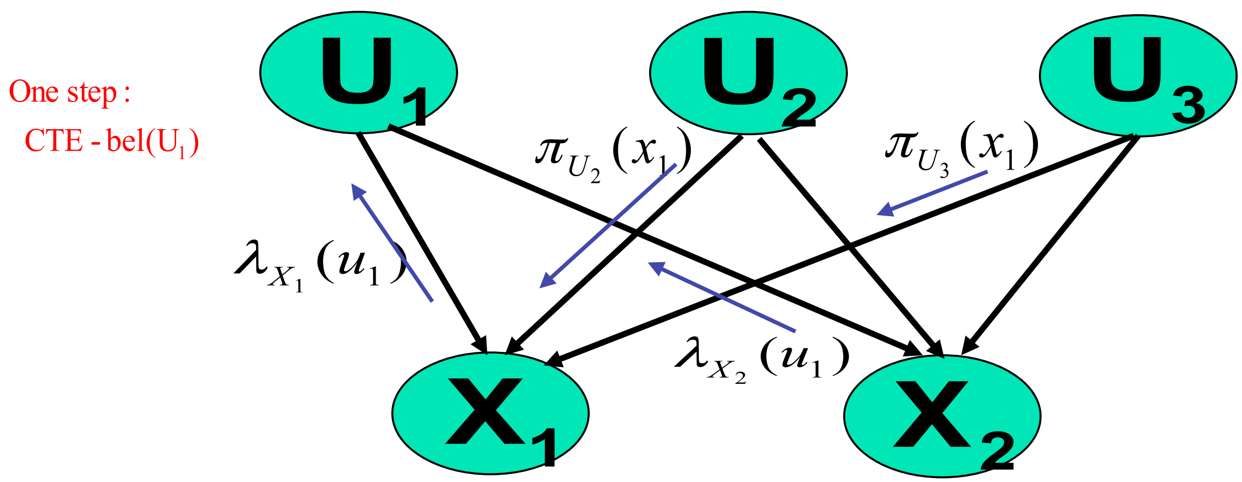
A polytree decomposition



- Running CTE = running Pearl's BP over the dual graph
- Dual-graph: nodes are cpts, arcs connect non-empty
- intersections. BP is Time and space linear

Iterative Belief Proapagation

- Belief propagation is exact for poly-trees
- IBP - applying BP iteratively to cyclic networks



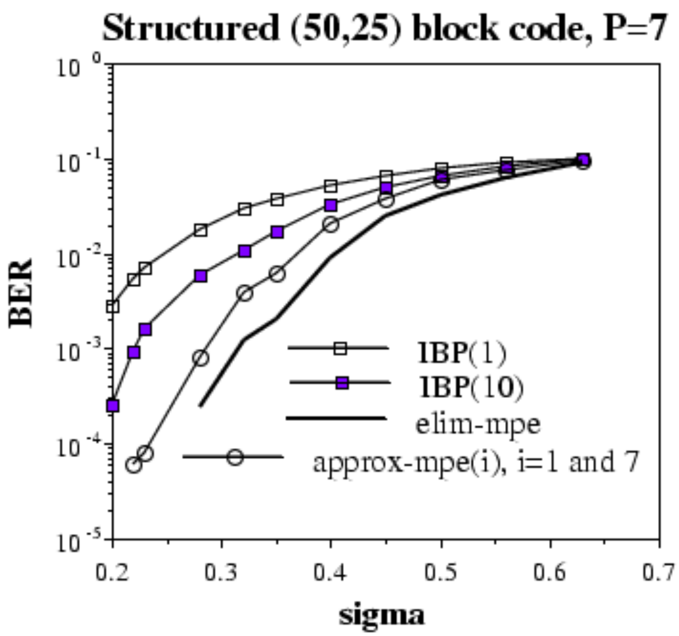
- No guarantees for convergence
- Works well for many coding networks

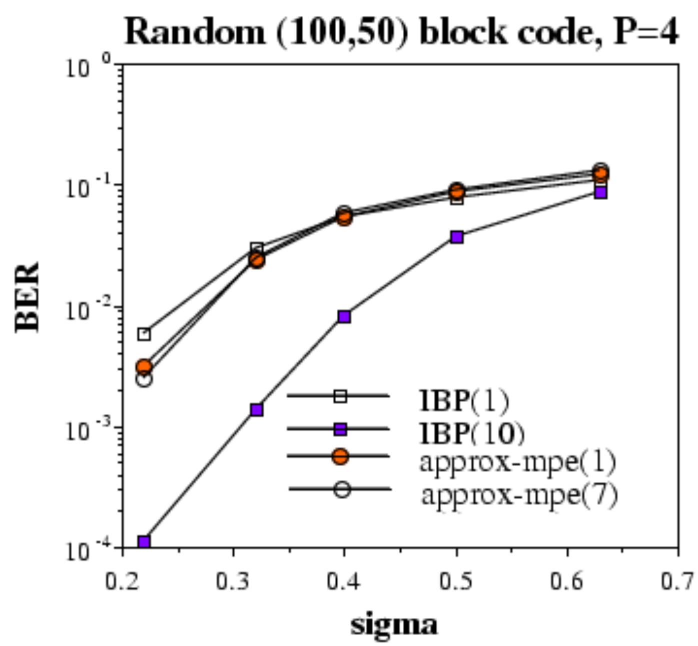
MBE-mpe vs. IBP

mbe - mpe is better on low - w * codes

IBP is better on randomly generated (high - w *) codes

Bit error rate (BER) as a function of noise (sigma):





Mini-buckets: summary

- Mini-buckets – local inference approximation
- Idea: bound size of recorded functions
- MBE-mpe(i) - mini-bucket algorithm for MPE
  - Better results for noisy-OR than for random problems
  - Accuracy increases with decreasing noise in coding
  - Accuracy increases for likely evidence
  - Sparser graphs -> higher accuracy
  - Coding networks: MBE-mpe outperforms IBP on low-induced width codes

Agenda

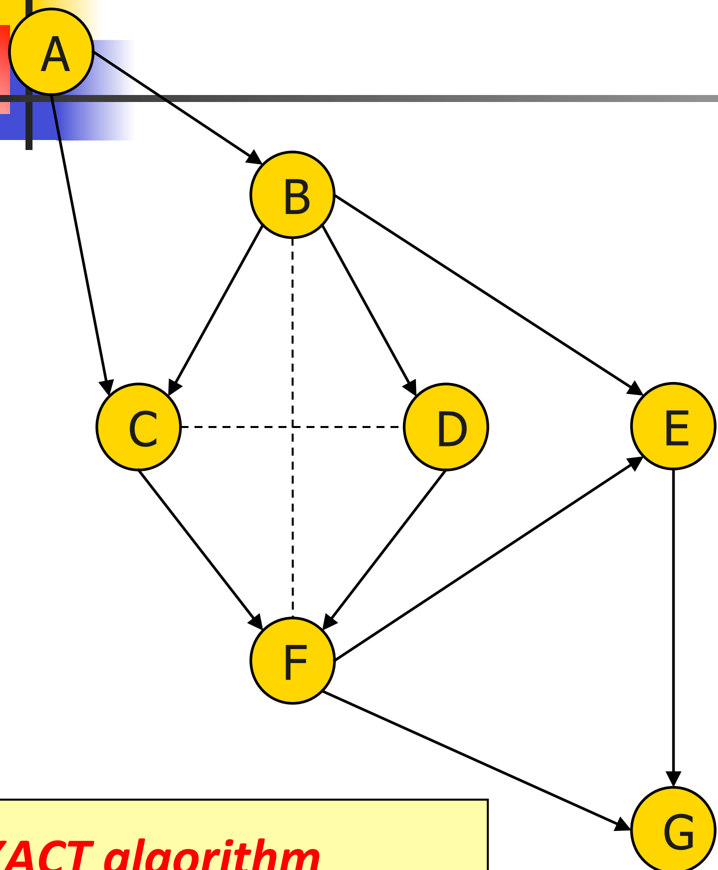
- Mini-bucket elimination
- Mini-clustering
- Iterative Belief propagation
- Iterative-join-graph propagation

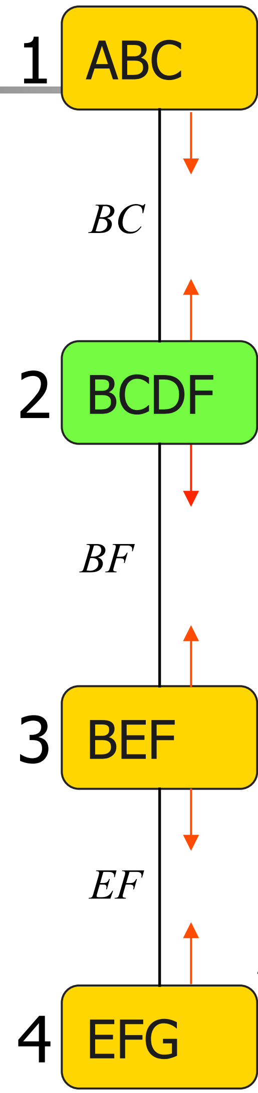
Cluster Tree Elimination - properties

- Correctness and completeness: Algorithm CTE is correct, i.e. it computes the exact joint probability of a single variable and the evidence.
- Time complexity: $O(\, deg \times (n+N) \times d^{\,w^*+1}\,)$
- Space complexity: $O(\, N \times d^{\,sep})$

where *deg* = the maximum degree of a node
n = number of variables (= number of CPTs)
N = number of nodes in the tree decomposition
d = the maximum domain size of a variable
*w** = the induced width
sep = the separator size

Join-Tree Clustering





$$h_{(1,2)}(b,c) = \sum_{a} p(a) \cdot p(b|a) \cdot p(c|a,b)$$

$$h_{(2,1)}(b,c) = \sum_{d,f} p(d|b) \cdot p(f|c,d) \cdot h_{(3,2)}(b,f)$$

$$h_{(2,3)}(b,f) = \sum_{c,d} p(d|b) \cdot p(f|c,d) \cdot h_{(1,2)}(b,c)$$

$$h_{(3,2)}(b,f) = \sum_{e} p(e|b,f) \cdot h_{(4,3)}(e,f)$$

$$h_{(3,4)}(e,f) = \sum_{b} p(e|b,f) \cdot h_{(2,3)}(b,f)$$

$$h_{(4,3)}(e,f) = p(G = g_e | e,f)$$

EXACT algorithm

Time and space:
exp(cluster size)=
exp(treewidth)

Mini-Clustering

Split a cluster into mini-clusters => bound complexity

$$\underbrace{\{h_1,\dots,h_r,h_{r+1},\dots,h_n\}}$$

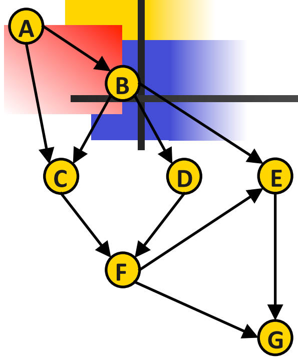
APPROXIMATE algorithm

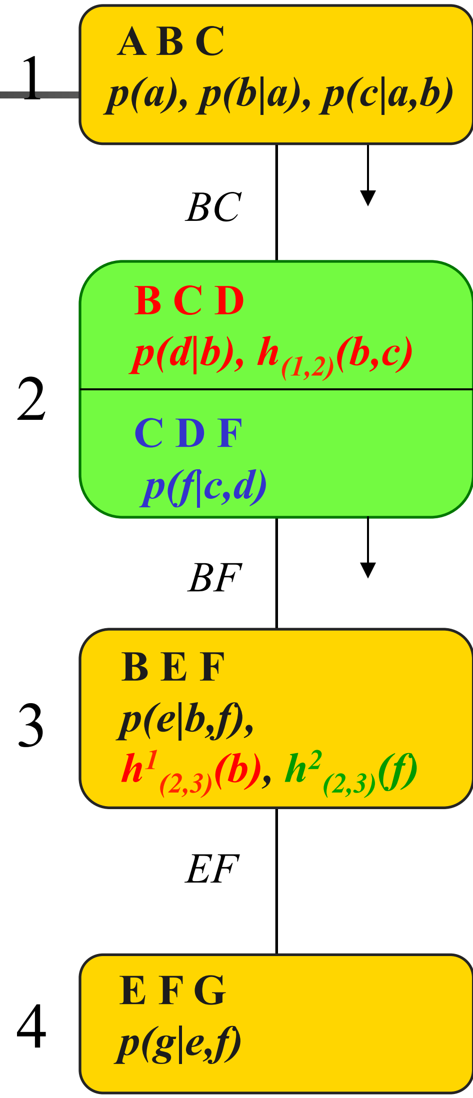
$$\underbrace{\{h_1,\dots,h_r\}} \qquad \underbrace{\{h_{r+1},\dots,h_n\}}$$

$$\sum_{elim}\prod_{i=1}^{n} h_i \quad \le \quad \left(\sum_{elim}\prod_{i=1}^{r} h_i\right)\cdot\left(\sum_{elim}\prod_{i=r+1}^{n} h_i\right)$$

Exponential complexity decrease $\quad O(e^n) \rightarrow O(e^{\mathrm{var}(r)}) + O(e^{\mathrm{var}(n-r)})$

Mini-Clustering, i-bound=3





$$h^1_{(1,2)}(b,c) = \sum_a p(a) \cdot p(b \mid a) \cdot p(c \mid a,b)$$

$$h^1_{(2,3)}(b) = \sum_{c,d} p(d \mid b) \cdot h^1_{(1,2)}(b,c)$$

$$h^2_{(2,3)}(f) = \max_{c,d} p(f \mid c,d)$$

APPROXIMATE algorithm

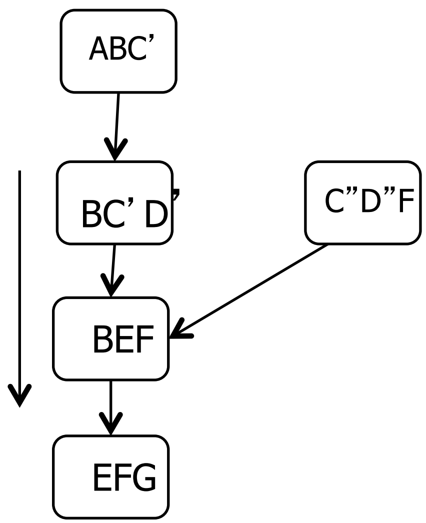
Time and space:
exp(i-bound)

$\downarrow$

Number of variables in a mini-cluster

Semantic of variable duplication for mini-clustering

We can have a different duplication of nodes going up and down. Example: going down (left) and up (right)



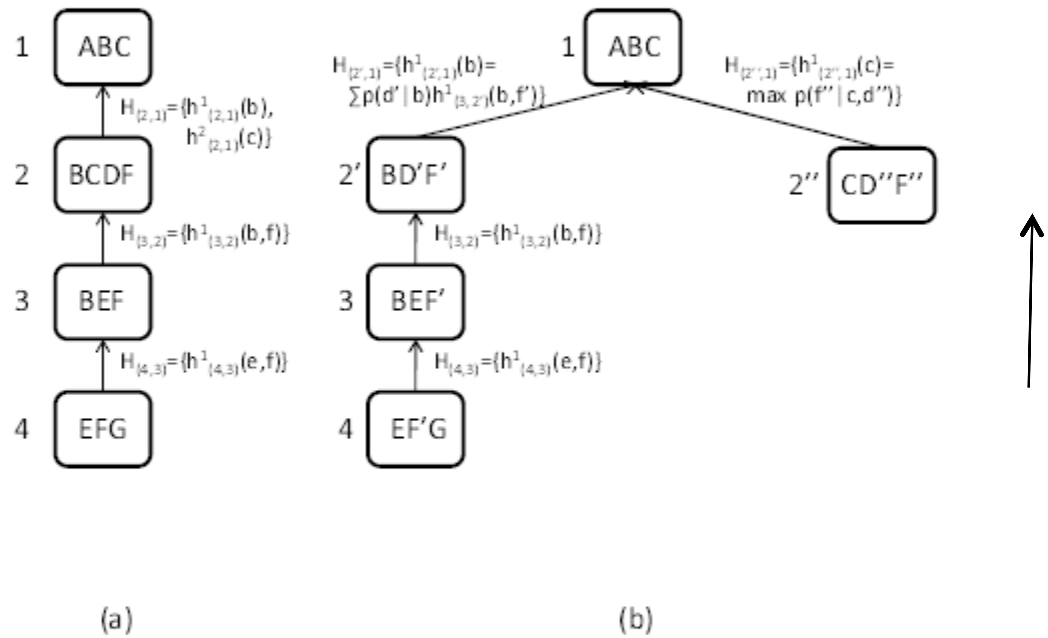


Figure 1.14: Node duplication semantics of MC: (a) trace of MC-BU(3); (b) trace of CTE-BU.

Mini-Clustering

- **Correctness and completeness**: Algorithm MC-bel(i) computes a bound (or an approximation) on the joint probability $P(X_i,e)$ of each variable and each of its values.
- **Time & space** complexity: $O(n \times hw^* \times k^{i})$

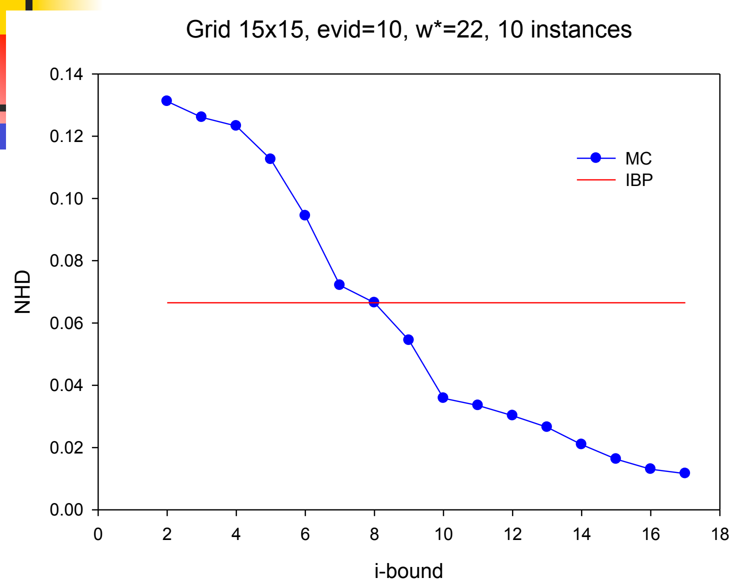
where $hw^* = max_u \mid \{f \mid f \cap \chi(u) \neq \phi\} \mid$

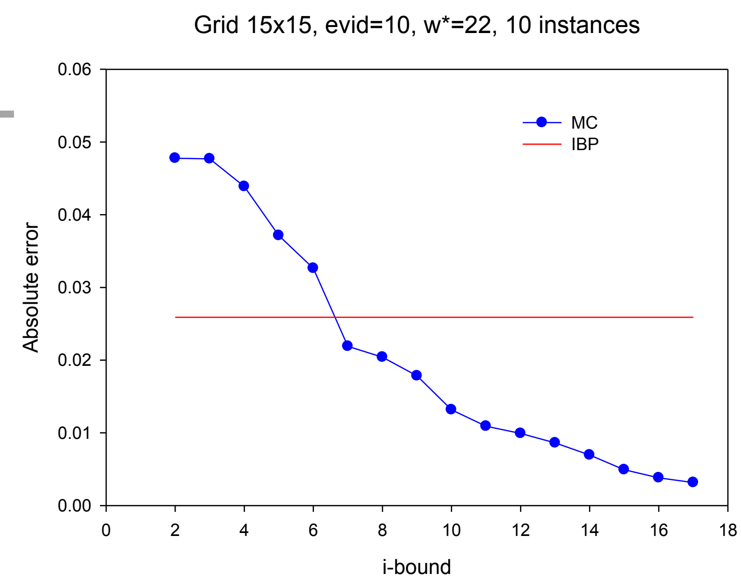
Lower bounds and mean approximations

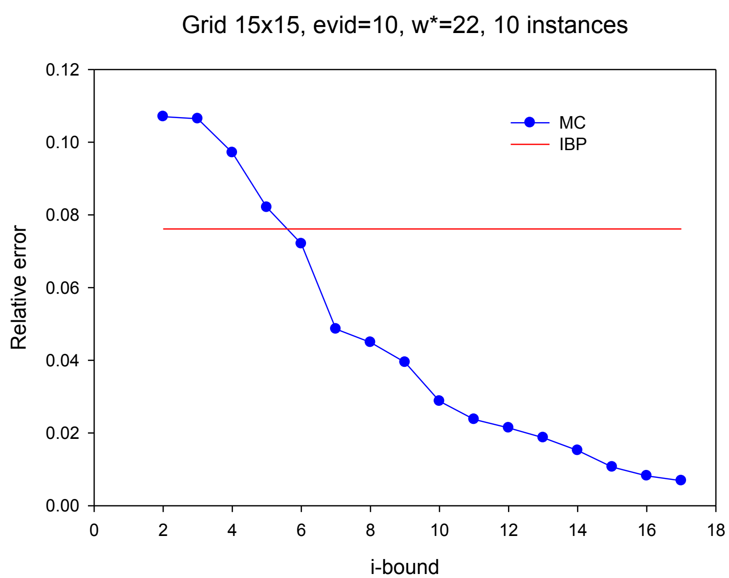
We can replace *max* operator by

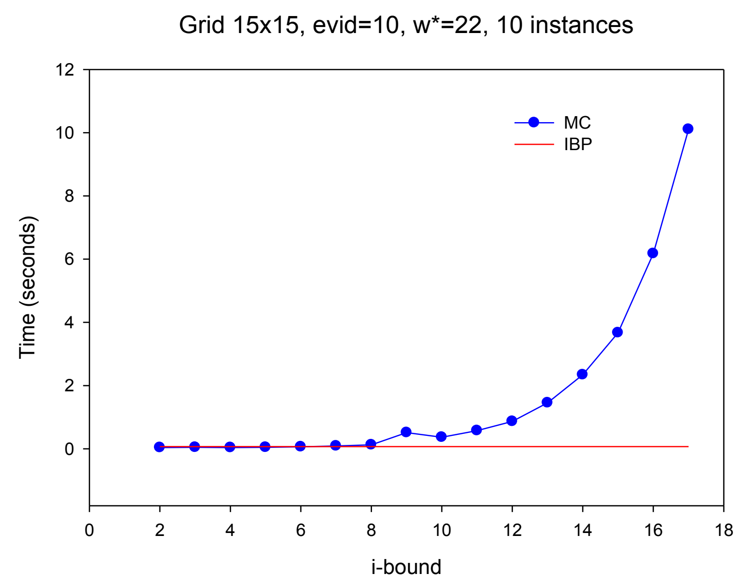
- *min* => lower bound on the joint
- *mean* => approximation of the joint

Grid 15x15 - 10 evidence

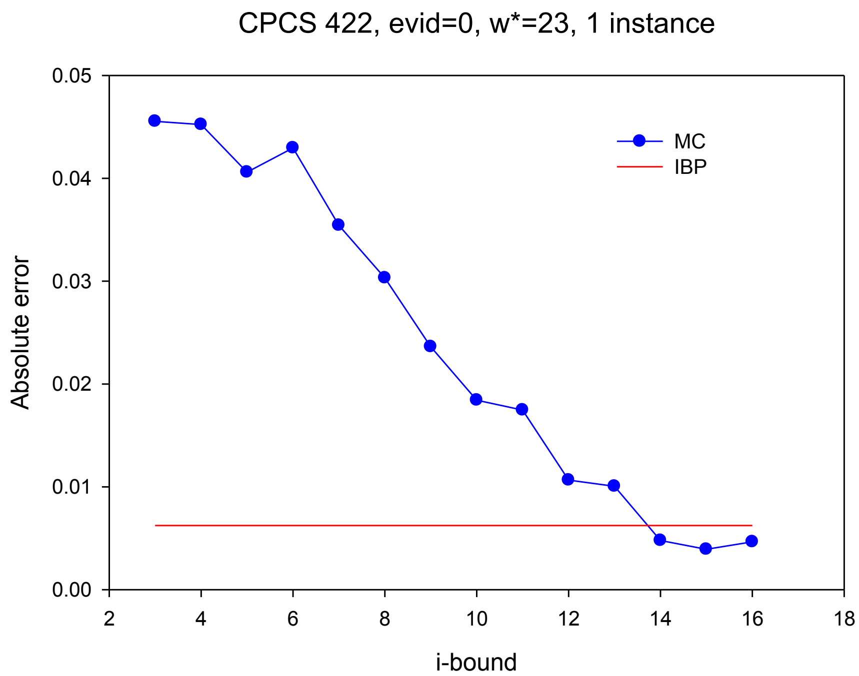


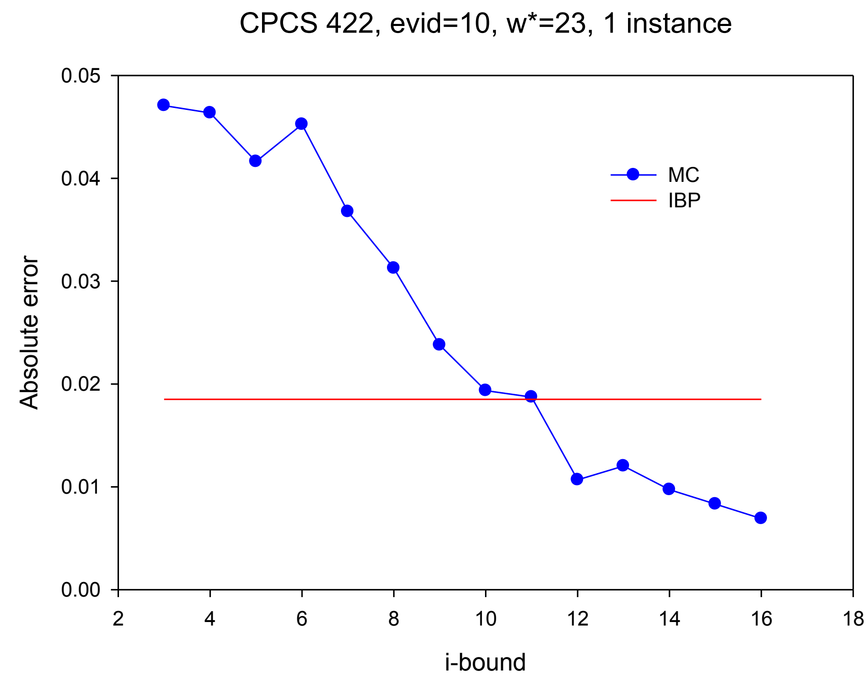






CPCS422 - Absolute error

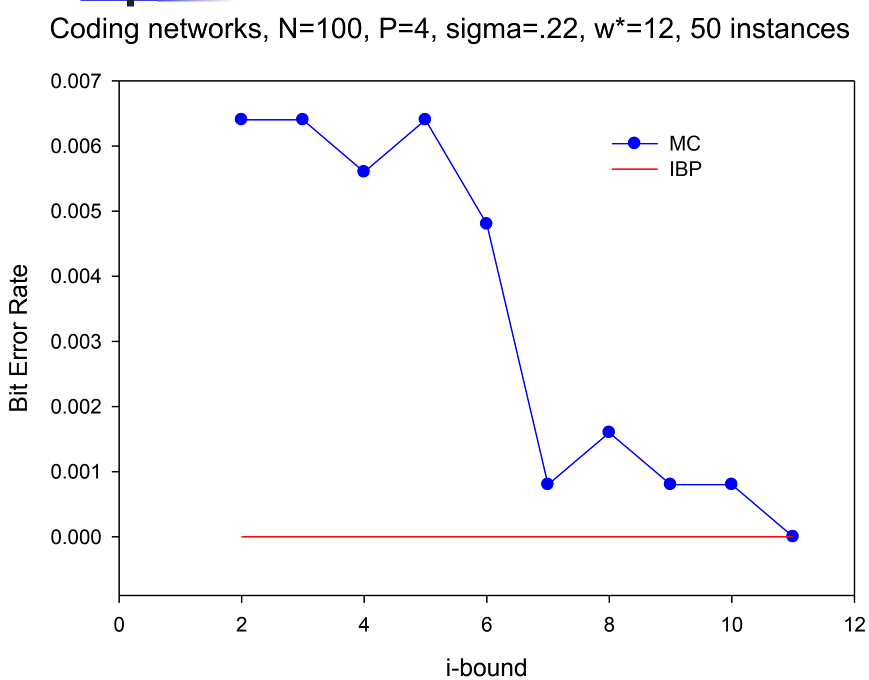


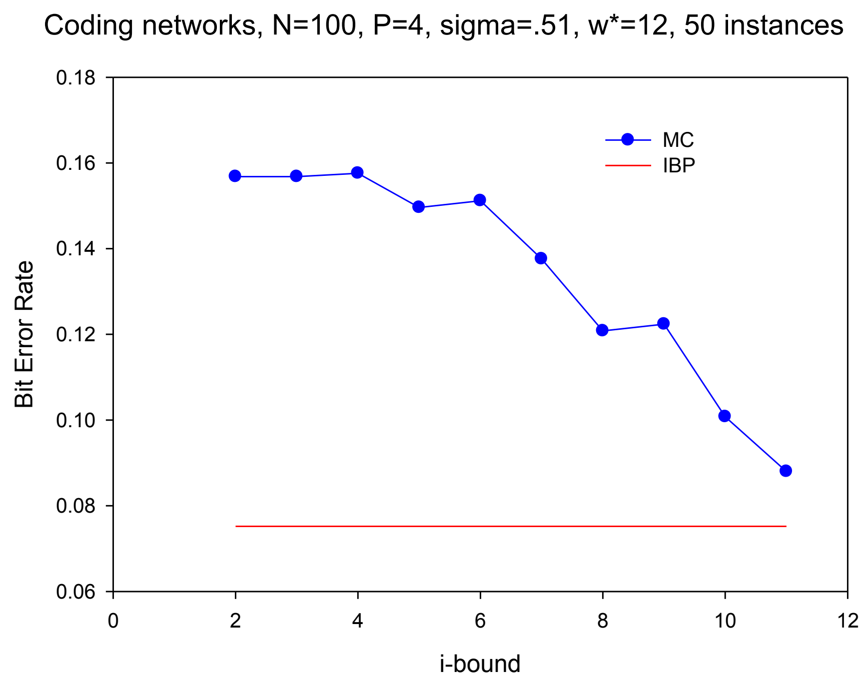


evidence=0

evidence=10

Coding networks - Bit Error Rate





sigma=0.22

sigma=.51

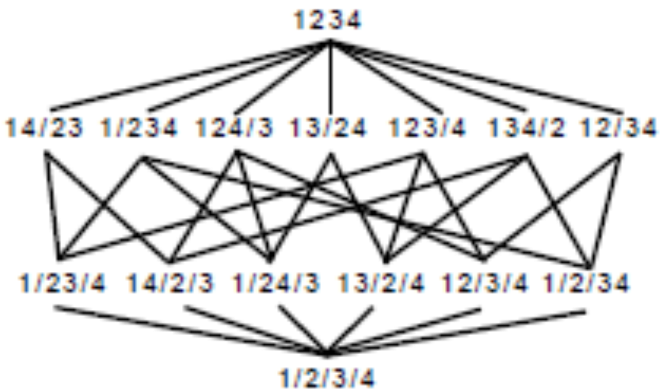
Heuristic for partitioning

Scope-based Partitioning Heuristic. The *scope-based* partition heuristic (SCP) aims at minimizing the number of mini-buckets in the partition by including in each minibucket as many functions as possible as long as the *i* bound is satisfied. First, single function mini-buckets are decreasingly ordered according to their arity. Then, each minibucket is absorbed into the left-most mini-bucket with whom it can be merged.

The time and space complexity of Partition(B, i) , where B is the partitioned bucket, using the SCP heuristic is $O(|B| \log(|B|) + |B|^2)$ and $O(exp(i))$, respectively.

The scope-based heuristic is is quite fast, its shortcoming is that it does not consider the actual information in the functions.

Content-based heuristics
(Rollon and Dechter 2010)



Partitioning lattice of bucket $\{f_1, f_2, f_3, f_4\}$.

- *Log relative error:*

$$RE(f,h) = \sum_t (\log(f(t)) - \log(h(t)))$$

- *Max log relative error:*

$$MRE(f,h) = \max_t \{\log(f(t)) - \log(h(t))\}$$

Use greedy heuristic derived from a distance function to decide which functions go into a single mini-bucket

Agenda

- Mini-bucket elimination
- Mini-clustering
- Iterative Belief propagation
- Iterative-join-graph propagation

Iterative Join Graph Propagation

- Loopy Belief Propagation
  - Cyclic graphs
  - Iterative
  - Converges fast in practice (no guarantees though)
  - Very good approximations (e.g., turbo decoding, LDPC codes, SAT – survey propagation)

- Mini-Clustering(i)
  - Tree decompositions
  - Only two sets of messages (inward, outward)
  - Anytime behavior – can improve with more time by increasing the i-bound

- We want to combine:
  - Iterative virtues of Loopy BP
  - Anytime behavior of Mini-Clustering(i)

IJGP - The basic idea

- Apply Cluster Tree Elimination to any *join-graph*
- We commit to graphs that are *I-maps*
- Avoid cycles as long as I-mapness is not violated
- Result: use *minimal arc-labeled* join-graphs

Minimal arc-labeled join-graph

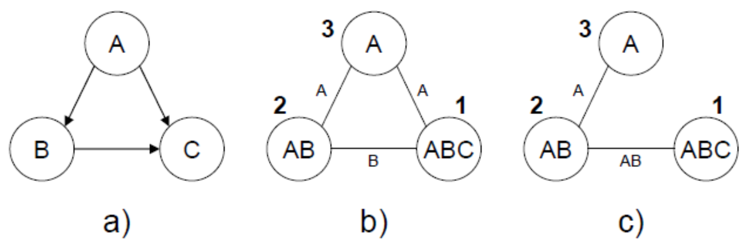


Figure 1.17: a) A belief network; b) A dual join-graph with singleton labels; c) A dual join-graph which is a join-tree

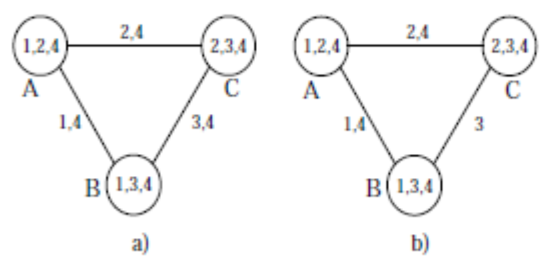
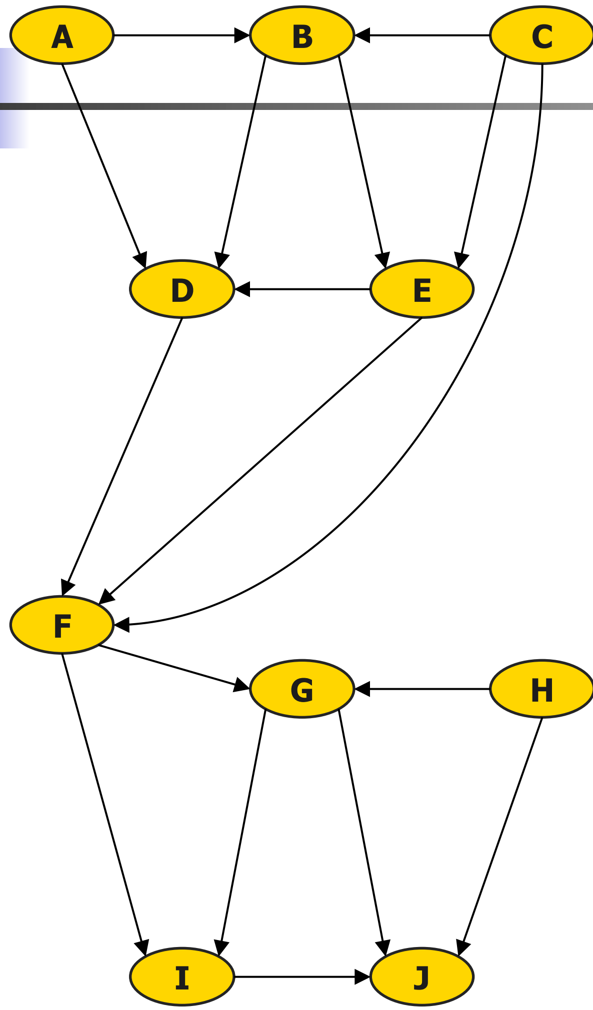
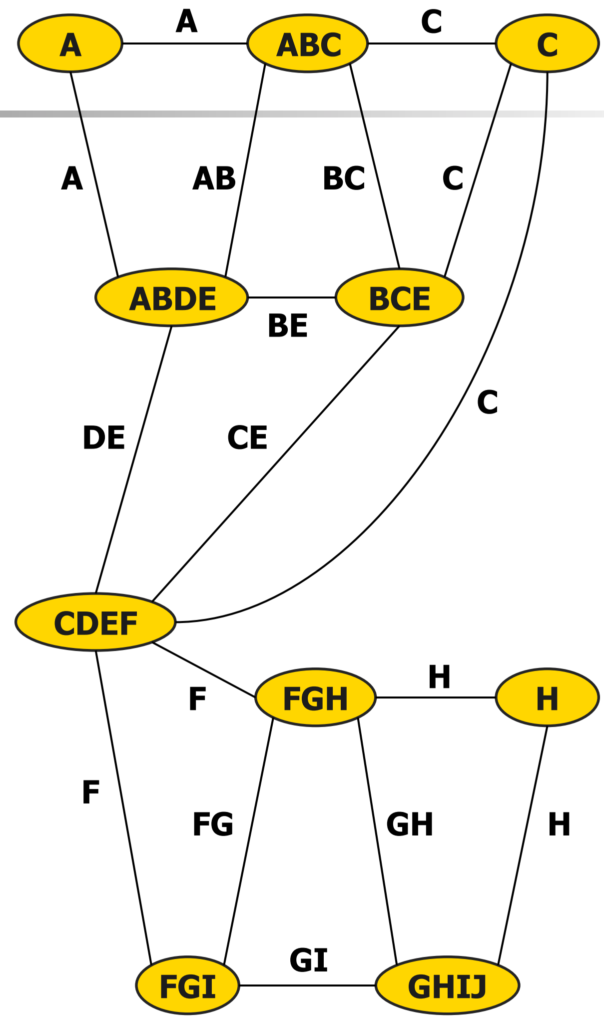


Figure 1.15: An arc-labeled decomposition

IJGP - Example



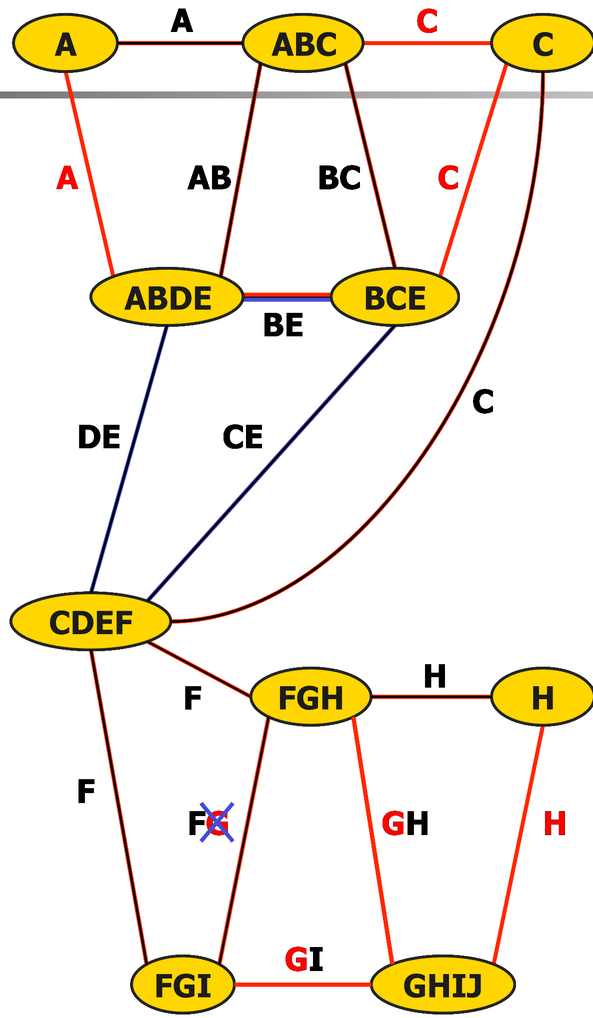
Belief network



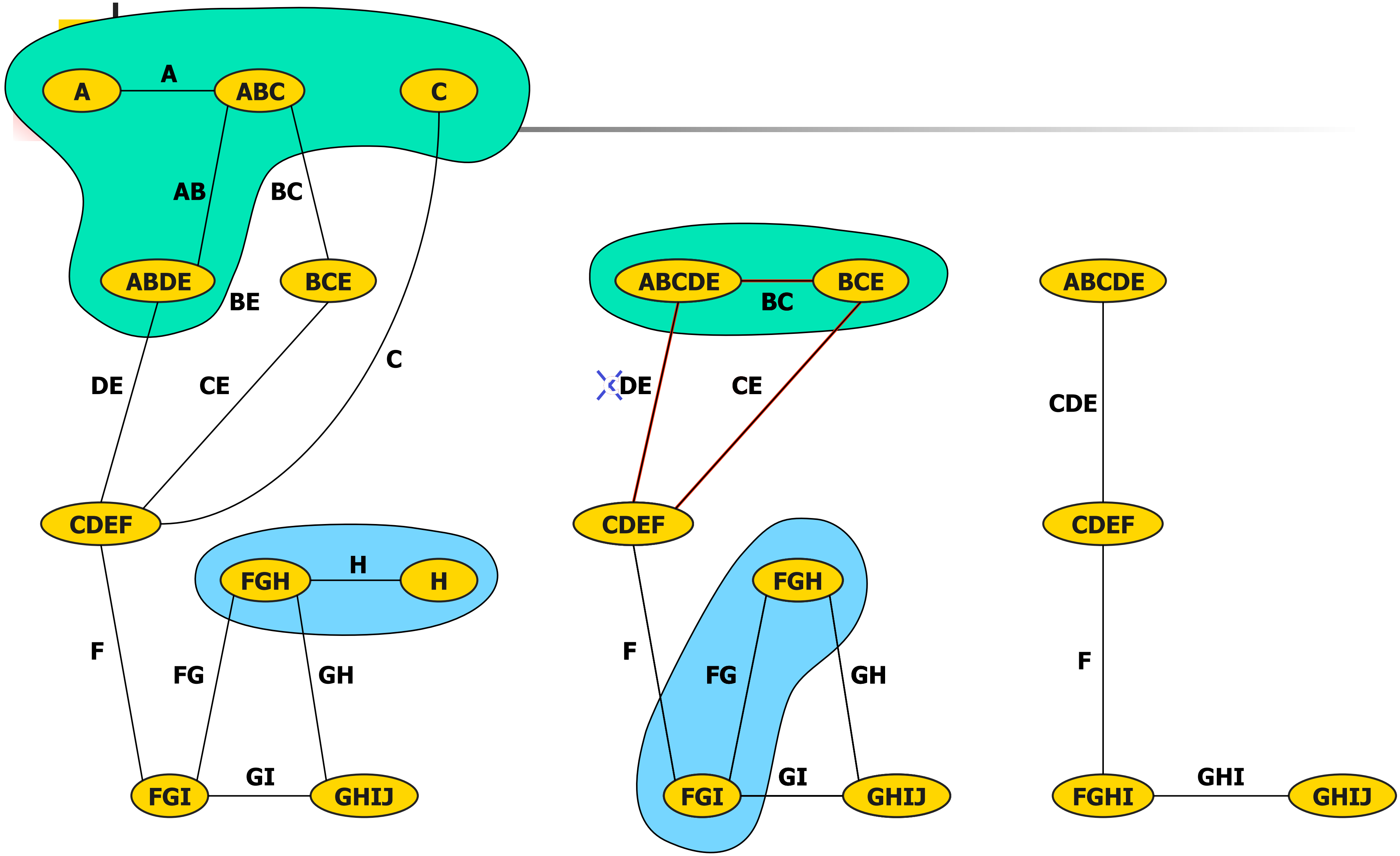
Loopy BP graph

Arc-Minimal Join-Graph

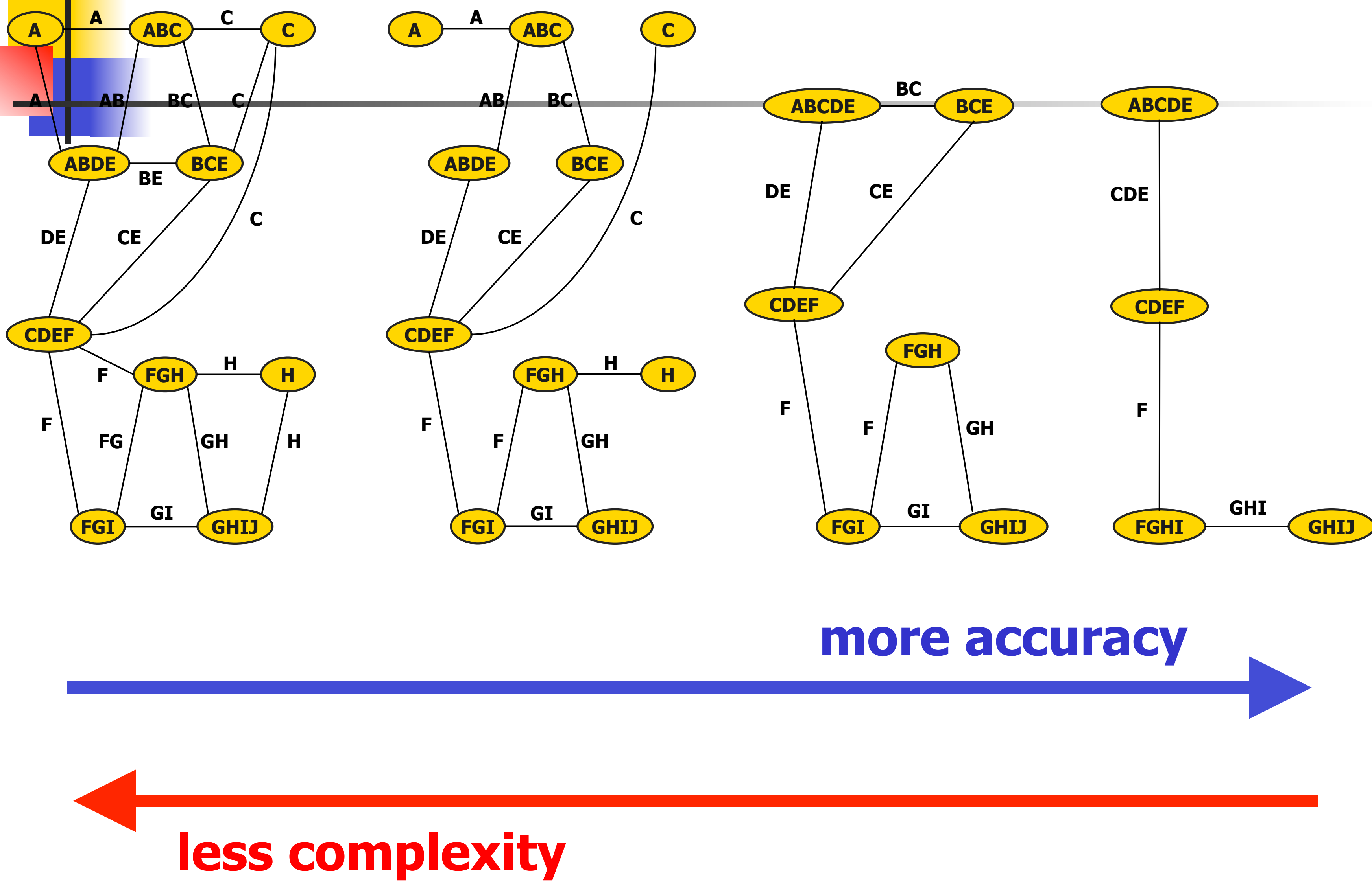
Arcs labeled with any single variable should form a TREE



Collapsing Clusters

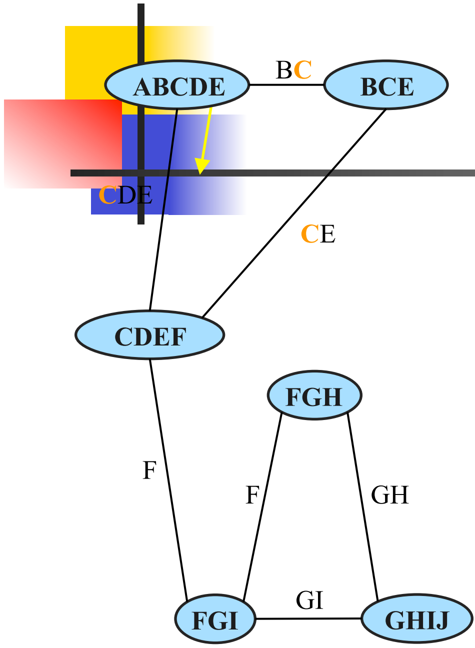


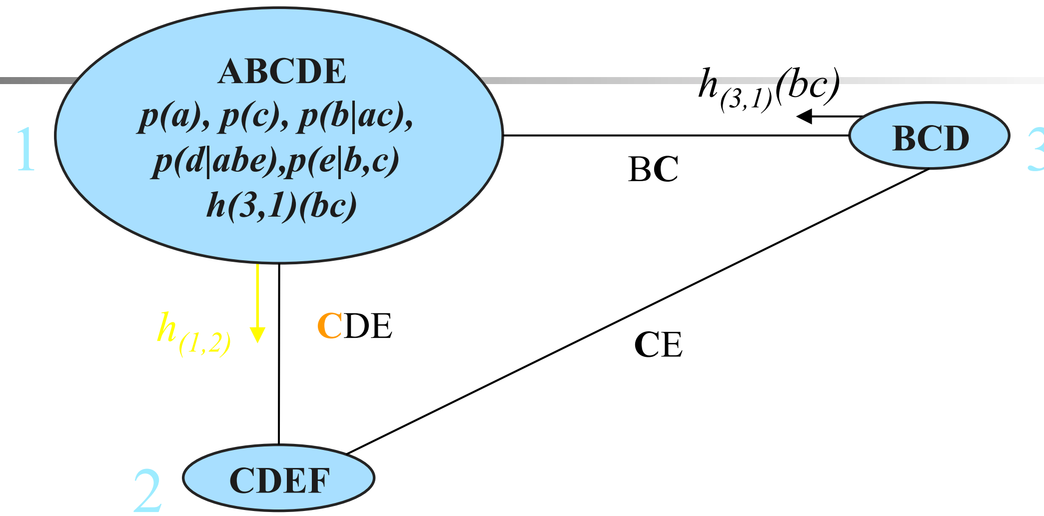
Join-Graphs



A
ABC
C
ABDE
BCE
CDEF
FGH
H
FGI
GHIJ
AB
BC
BE
DE
CE
F
FG
GH
GI
ABCDE
CDE
FGHI
GHI
more accuracy
less complexity

Message propagation





Minimal arc-labeled:
sep(1,2)={D,E}
elim(1,2)={A,B,C}

$$h_{(1,2)}(de) = \sum_{a,b,c} p(a)p(c)p(b \mid ac)p(d \mid abe)p(e \mid bc)h_{(3,1)}(bc)$$

Non-minimal arc-labeled:
sep(1,2)={C,D,E}
elim(1,2)={A,B}

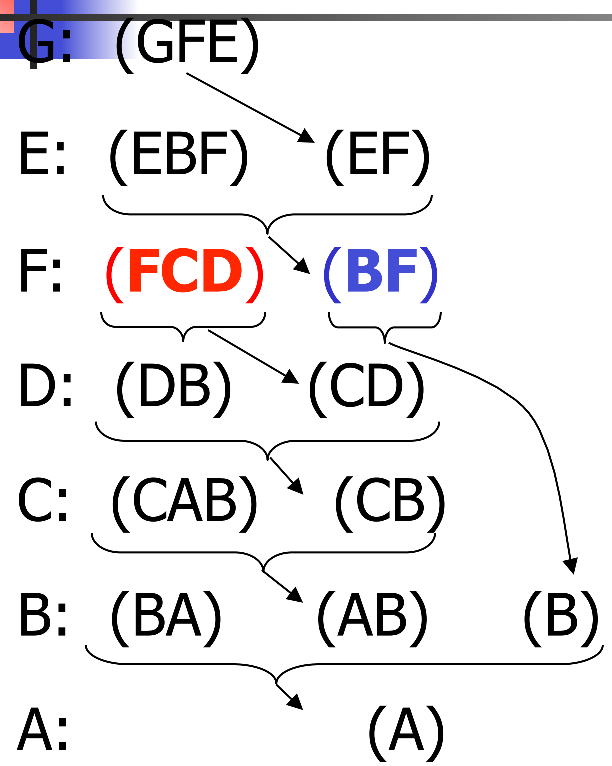
$$h_{(1,2)}(cde) = \sum_{a,b} p(a)p(c)p(b \mid ac)p(d \mid abe)p(e \mid bc)h_{(3,1)}(bc)$$

Bounded decompositions

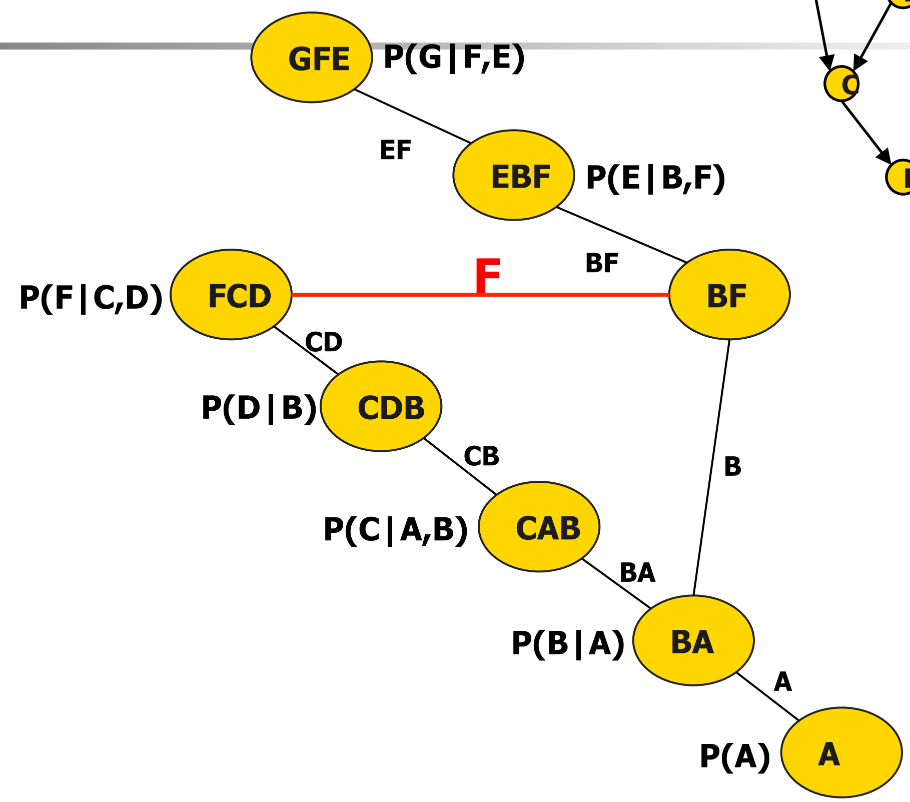
- We want arc-labeled decompositions such that:
  - the cluster size (internal width) is bounded by *i* (the accuracy parameter)
  - the width of the decomposition as a graph (external width) is as small as possible

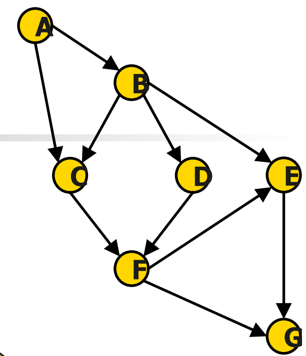
- Possible approaches to build decompositions:
  - partition-based algorithms - inspired by the mini-bucket decomposition
  - grouping-based algorithms

Constructing Join-Graphs



a) schematic mini-bucket(i), i=3





b) arc-labeled join-graph decomposition

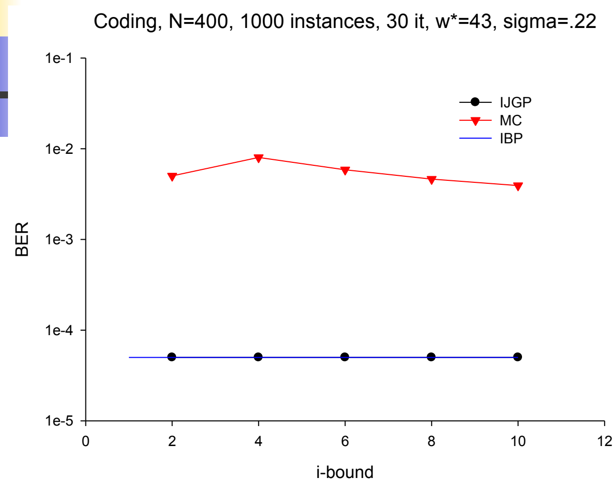
IJGP properties

- IJGP(*i*) applies BP to min arc-labeled join-graph, whose cluster size is bounded by *i*
- On join-trees IJGP finds exact beliefs
- IJGP is a Generalized Belief Propagation algorithm (Yedidia, Freeman, Weiss 2001)
- Complexity of one iteration:
  - time: $O(deg \bullet (n+N) \bullet d^{i+1})$
  - space: $O(N \bullet d^{\theta})$

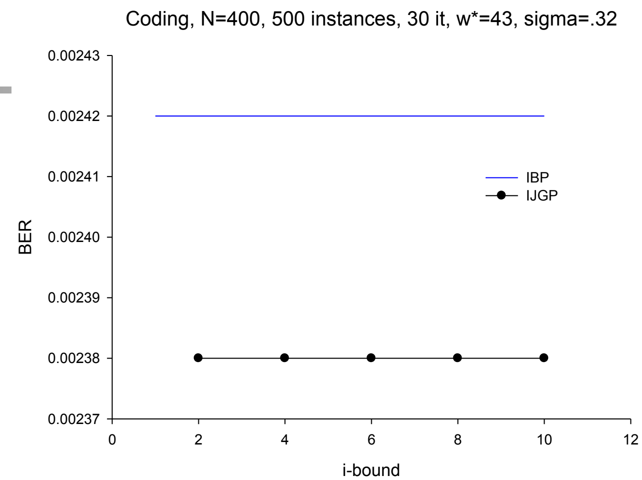
Empirical evaluation

- Algorithms:
  - Exact
  - IBP
  - MC
  - IJGP
- Measures:
  - Absolute error
  - Relative error
  - Kulbach-Leibler (KL) distance
  - Bit Error Rate
  - Time
- Networks (all variables are binary):
  - Random networks
  - Grid networks (MxM)
  - CPCS 54, 360, 422
  - Coding networks

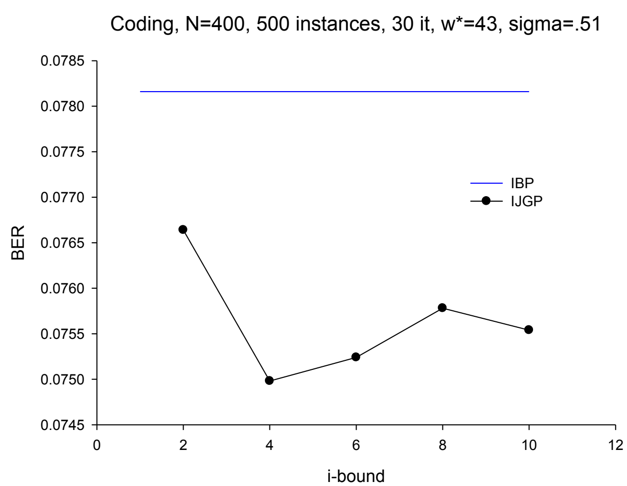
Coding networks - BER



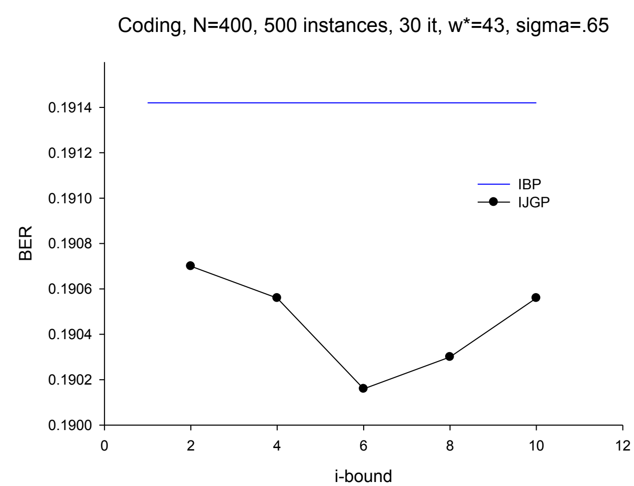
sigma=.22



sigma=.32

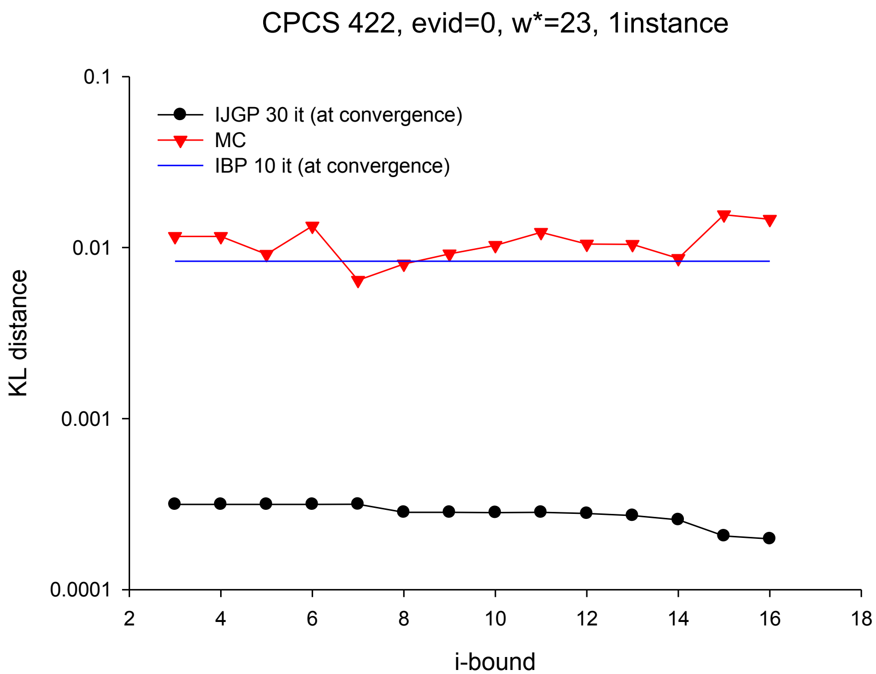


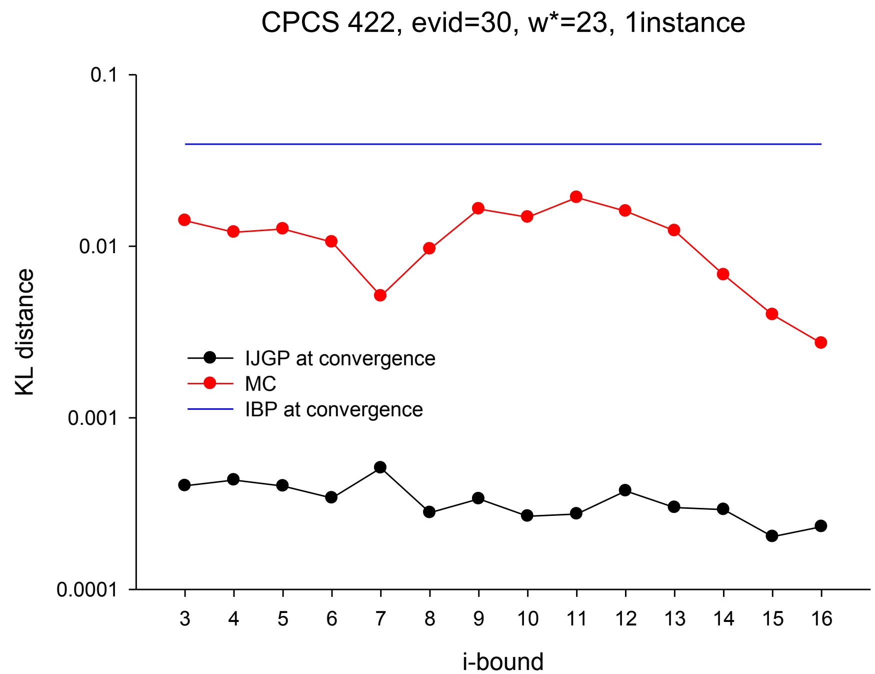
sigma=.51



sigma=.65

CPCS 422 – KL Distance

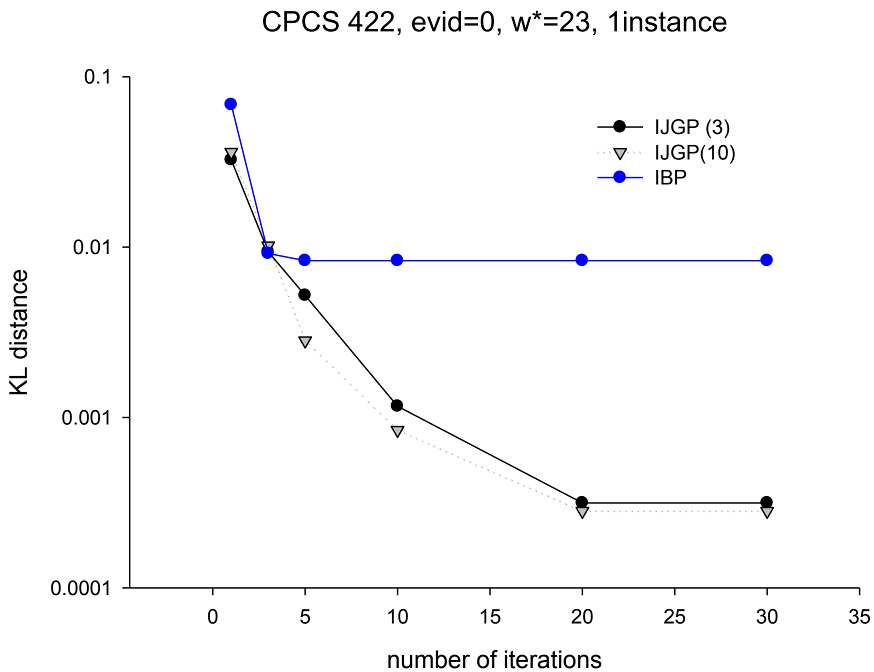




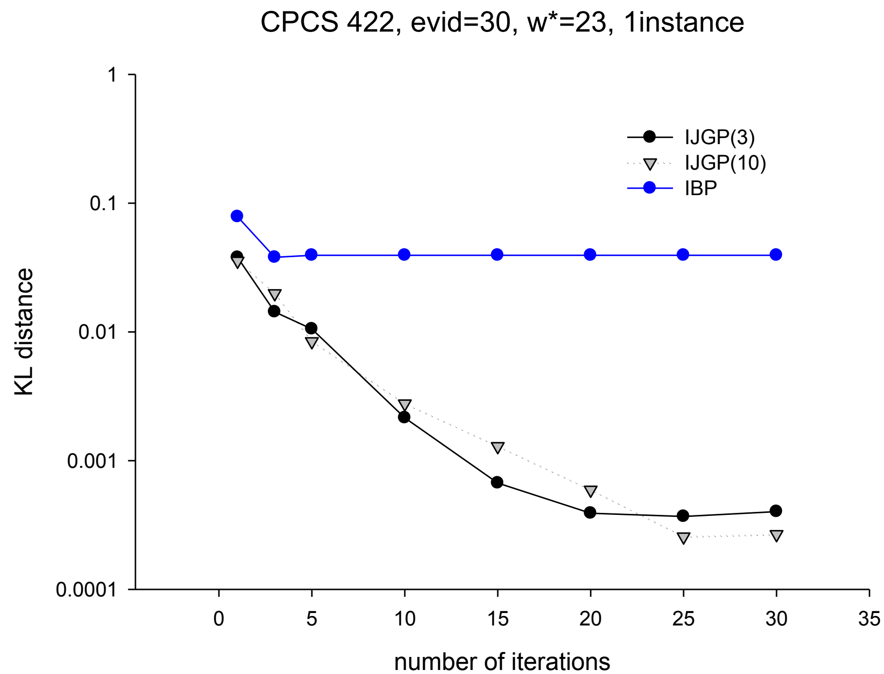
evidence=0

evidence=30

CPCS 422 – KL vs. Iterations

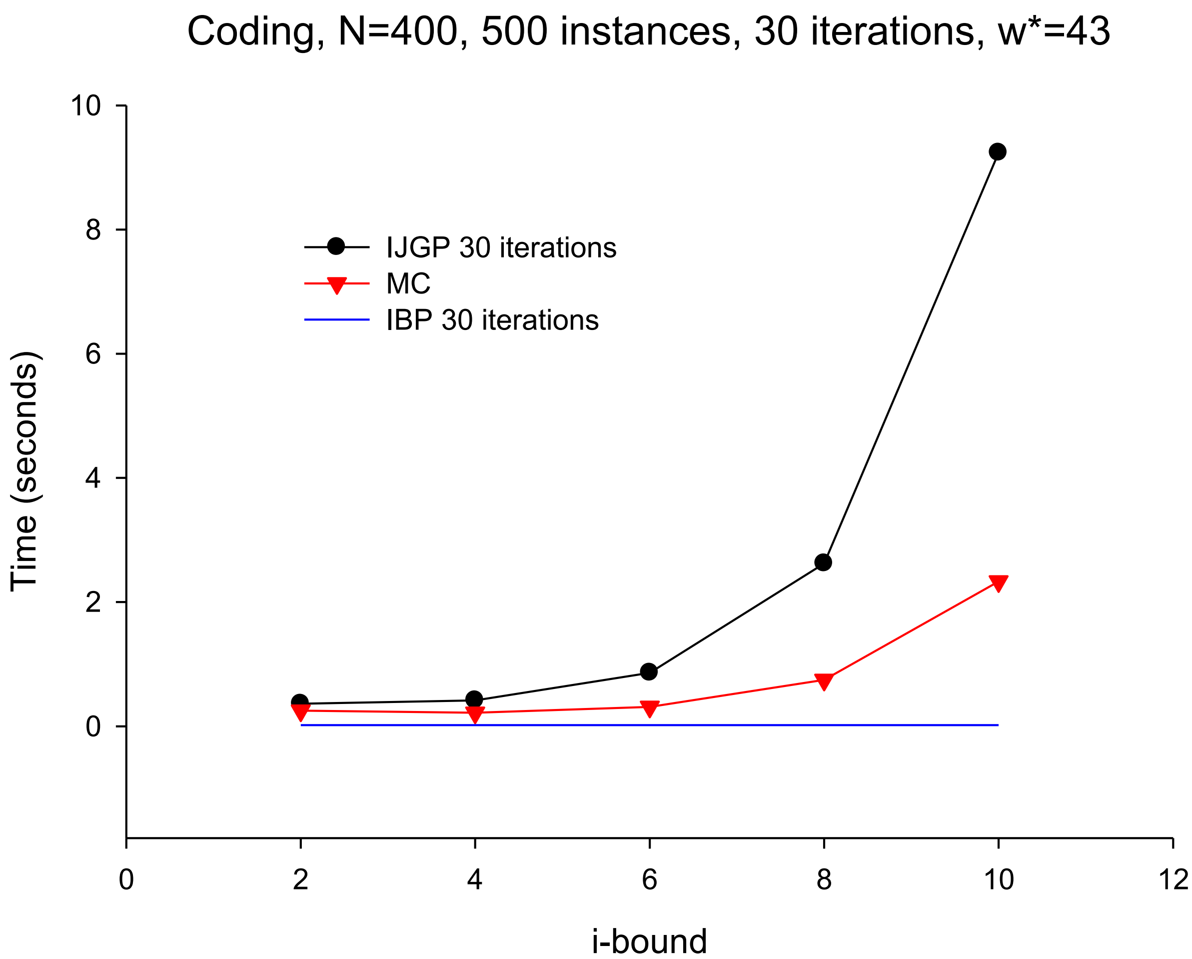


evidence=0



evidence=30

Coding networks - Time



More On the Power of Belief Propagation

- BP as local minima of KL distance
- BP's power from constraint propagation perspective.

More On the Power of Belief Propagation

- BP as local minima of KL distance
- BP's power from constraint propagation perspective.

The Kullback-Leibler Divergence

The Kullback-Leibler divergence (KL–divergence)

$$\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e})) = \sum_{\mathbf{x}} \Pr'(\mathbf{x}|\mathbf{e}) \log \frac{\Pr'(\mathbf{x}|\mathbf{e})}{\Pr(\mathbf{x}|\mathbf{e})}$$

- $\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e}))$ is non-negative
- equal to zero if and only if $\Pr'(\mathbf{X}|\mathbf{e})$ and $\Pr(\mathbf{X}|\mathbf{e})$ are equivalent.

The Kullback-Leibler Divergence

KL–divergence is not a true distance measure in that it is not symmetric. In general:

$$\mathrm{KL}(\mathrm{Pr}'(\mathbf{X}|\mathbf{e}), \mathrm{Pr}(\mathbf{X}|\mathbf{e})) \neq \mathrm{KL}(\mathrm{Pr}(\mathbf{X}|\mathbf{e}), \mathrm{Pr}'(\mathbf{X}|\mathbf{e})).$$

- $\mathrm{KL}(\mathrm{Pr}'(\mathbf{X}|\mathbf{e}), \mathrm{Pr}(\mathbf{X}|\mathbf{e}))$ weighting the KL–divergence by the approximate distribution Pr'
- We shall indeed focus on the KL–divergence weighted by the approximate distribution as it has some useful computational properties.

The Kullback-Leibler Divergence

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$

The KL–divergence between $\Pr$ and another distribution $\Pr'$ can be written as a sum of three components:

$$
\begin{aligned}
&\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e})) \\
&\quad = \quad -\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) - \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}}) + \log \Pr(\mathbf{e}),
\end{aligned}
$$

where

- $\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) = -\sum_{\mathbf{x}} \Pr'(\mathbf{x}|\mathbf{e}) \log \Pr'(\mathbf{x}|\mathbf{e})$ is the entropy of the conditioned approximate distribution $\Pr'(\mathbf{X}|\mathbf{e})$.
- $\mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}}) = \sum_{x\mathbf{u}} \Pr'(x\mathbf{u}|\mathbf{e}) \log \lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}}$ is a set of expectations over the original network parameters weighted by the conditioned approximate distribution.

The Kullback-Leibler Divergence

A distribution $\Pr'(\mathbf{X}|\mathbf{e})$ minimizes the KL-divergence $\mathrm{KL}(\Pr'(\mathbf{X}|\mathbf{e}), \Pr(\mathbf{X}|\mathbf{e}))$ if it maximizes

$$\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}})$$

Competing properties of $\Pr'(\mathbf{X}|\mathbf{e})$ that minimize the KL–divergence:

- $\Pr'(\mathbf{X}|\mathbf{e})$ should match the original distribution by giving more weight to more likely parameters $\lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}}$ (i.e, maximize the expectations).
- $\Pr'(\mathbf{X}|\mathbf{e})$ should not favor unnecessarily one network instantiation over another by being evenly distributed (i.e., maximize the entropy).

Optimizing the KL-Divergence

The approximations computed by IBP are based on assuming an approximate distribution $\Pr'(\mathbf{X})$ that factors as follows:

$$\Pr'(\mathbf{X}|\mathbf{e}) = \prod_{X\mathbf{U}} \frac{\Pr'(X\mathbf{U}|\mathbf{e})}{\prod_{U \in \mathbf{U}} \Pr'(U|\mathbf{e})}$$

- This choice of $\Pr'(\mathbf{X}|\mathbf{e})$ is expressive enough to describe distributions $\Pr(\mathbf{X}|\mathbf{e})$ induced by polytree networks $\mathcal{N}$
- In the case where $\mathcal{N}$ is not a polytree, then we are simply trying to fit $\Pr(\mathbf{X}|\mathbf{e})$ into an approximation $\Pr'(\mathbf{X}|\mathbf{e})$ as if it were generated by a polytree network.
- The entropy of distribution $\Pr'(\mathbf{X}|\mathbf{e})$ can be expressed as:

$$\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) = -\sum_{X\mathbf{U}} \sum_{x\mathbf{u}} \Pr'(x\mathbf{u}|\mathbf{e}) \log \frac{\Pr'(x\mathbf{u}|\mathbf{e})}{\prod_{u \sim \mathbf{u}} \Pr'(u|\mathbf{e})}$$

Optimizing the KL-Divergence

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$. Then IBP messages are a fixed point if and only if IBP marginals $\mu_u = BEL(u)$ and $\mu_{x\mathbf{u}} = BEL(x\mathbf{u})$ are a stationary point of:

$$
\begin{aligned}
&\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{X\mathbf{U}} \mathrm{AVG}'(\log \lambda_{\mathbf{e}}(X)\Theta_{X|\mathbf{U}}) \\
&= \quad -\sum_{X\mathbf{U}}\sum_{x\mathbf{u}} \mu_{x\mathbf{u}} \log \frac{\mu_{x\mathbf{u}}}{\prod_{u\sim\mathbf{u}} \mu_u} + \sum_{X\mathbf{U}}\sum_{x\mathbf{u}} \mu_{x\mathbf{u}} \log \lambda_{\mathbf{e}}(x)\theta_{x|\mathbf{u}},
\end{aligned}
$$

under normalization constraints:

$$
\sum_u \mu_u = \sum_{x\mathbf{u}} \mu_{x\mathbf{u}} = 1
$$

for each family $X\mathbf{U}$ and parent U, and under consistency constraints:

$$
\sum_{x\mathbf{u}\sim y} \mu_{x\mathbf{u}} = \mu_y
$$

for each family instantiation $x\mathbf{u}$ and value y of family member $Y \in X\mathbf{U}$.

Optimizing the KL-Divergence

- IBP fixed points are stationary points of the KL–divergence: they may only be local minima, or they may not be minima.
- When IBP performs well, it will often have fixed points that are indeed minima of the KL–divergence.
- For problems where IBP does not behave as well, we will next seek approximations $\Pr'$ whose factorizations are more expressive than that of the polytree-based factorization.

Generalized Belief Propagation

If a distribution $\Pr'$ has the form:

$$\Pr'(\mathbf{X}|\mathbf{e}) = \frac{\prod_{\mathbf{C}} \Pr'(\mathbf{C}|\mathbf{e})}{\prod_{\mathbf{S}} \Pr'(\mathbf{S}|\mathbf{e})},$$

then its entropy has the form:

$$\mathrm{ENT}'(\mathbf{X}|\mathbf{e}) = \sum_{\mathbf{C}} \mathrm{ENT}'(\mathbf{C}|\mathbf{e}) - \sum_{\mathbf{S}} \mathrm{ENT}'(\mathbf{S}|\mathbf{e}).$$

When the marginals $\Pr'(\mathbf{C}|\mathbf{e})$ and $\Pr'(\mathbf{S}|\mathbf{e})$ are readily available, the ENT component of the KL–divergence can be computed efficiently.

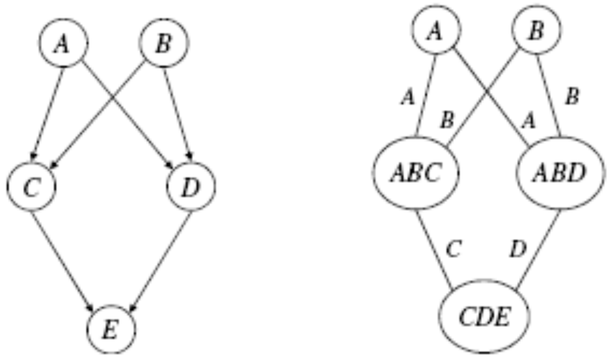
Joingraphs

While a jointree induces an exact factorization of a distribution, a joingraph G induces an approximate factorization:

$$\mathrm{Pr}'(\mathbf{X}|\mathbf{e}) = \frac{\prod_i \mathrm{Pr}'(\mathbf{C}_i|\mathbf{e})}{\prod_{ij} \mathrm{Pr}'(\mathbf{S}_{ij}|\mathbf{e})}$$

which is a product of cluster marginals over a product of separator marginals. When the joingraph corresponds to a jointree, the above factorization will be exact.

Joingraphs

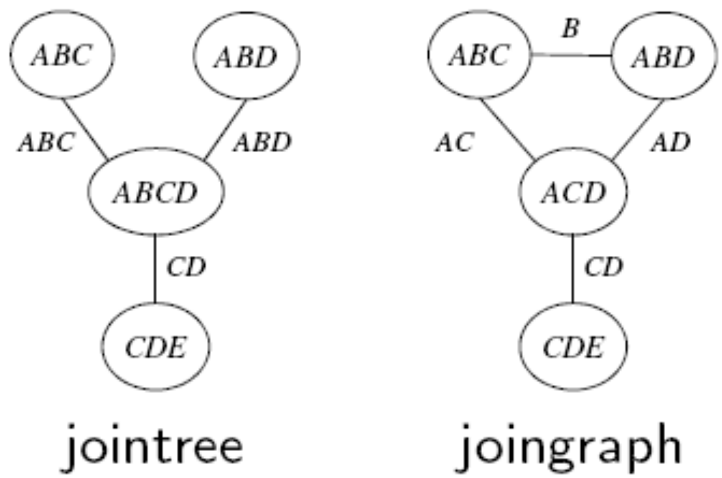


Bayesian network

dual joingraph

A dual joingraph leads to the factorization used by IBP.

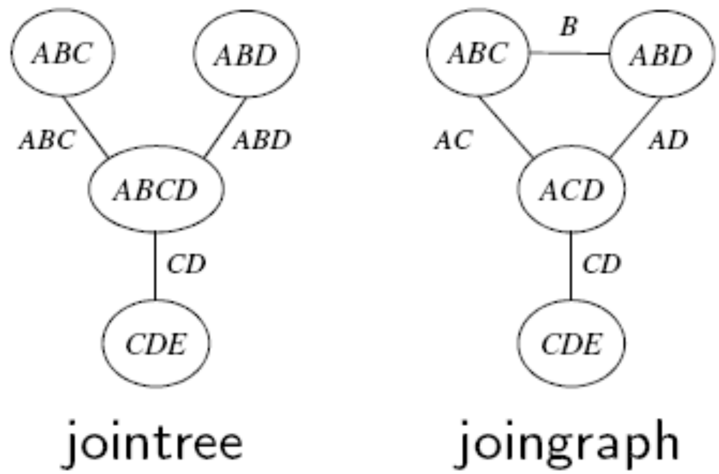
Joingraphs



The jointree induces the following factorization, which is exact:

$$\mathrm{Pr}'(\mathbf{X}|\mathbf{e}) = \frac{\mathrm{Pr}'(ABC|\mathbf{e})\mathrm{Pr}'(ABD|\mathbf{e})\mathrm{Pr}'(ABCD|\mathbf{e})\mathrm{Pr}'(CDE|\mathbf{e})}{\mathrm{Pr}'(ABC|\mathbf{e})\mathrm{Pr}'(ABD|\mathbf{e})\mathrm{Pr}'(CD|\mathbf{e})}$$

Joingraphs



The joingraph induces the following factorization:

$$\Pr'(\mathbf{X}|\mathbf{e}) = \frac{\Pr'(ABC|\mathbf{e})\Pr'(ABD|\mathbf{e})\Pr'(ACD|\mathbf{e})\Pr'(CDE|\mathbf{e})}{\Pr'(B|\mathbf{e})\Pr'(AC|\mathbf{e})\Pr'(AD|\mathbf{e})\Pr'(CD|\mathbf{e})}$$

terative Joingraph Propagation

Computing cluster marginals $\mu_{\mathbf{c}_i} = \Pr'(\mathbf{c}_i|\mathbf{e})$ and separator marginals $\mu_{\mathbf{s}_{ij}} = \Pr'(\mathbf{s}_{ij}|\mathbf{e})$ that minimize the KL–divergence between $\Pr'(\mathbf{X}|\mathbf{e})$ and $\Pr(\mathbf{X}|\mathbf{e})$

This optimization problem can be solved using a generalization of IBP, called iterative joingraph propagation (IJGP), which is a message passing algorithm that operates on a joingraph.

Iterative Joingraph Propagation

IJGP(G, Φ)

input:

G: a joingraph

Φ: factors assigned to clusters of G

output: approximate marginal $BEL(\mathbf{C}_i)$ for each node i in the joingraph G.

main:

```
t ← 0
initialize all messages M_ij^t (uniformly)
while messages have not converged do
    t ← t + 1
    for each joingraph edge i–j do
        M_ij^t ← η Σ_{C_i \ S_ij} Φ_i Π_{k≠j} M_ki^{t−1}
        M_ji^t ← η Σ_{C_j \ S_ij} Φ_j Π_{k≠i} M_kj^{t−1}
    end for
end while
return BEL(C_i) ← η Φ_i Π_k M_ki^t for each node i
```

$$M_{ij}^{t} \leftarrow \eta \sum_{\mathbf{C}_i \setminus \mathbf{S}_{ij}} \Phi_i \prod_{k \neq j} M_{ki}^{t-1}$$

$$M_{ji}^{t} \leftarrow \eta \sum_{\mathbf{C}_j \setminus \mathbf{S}_{ij}} \Phi_j \prod_{k \neq i} M_{kj}^{t-1}$$

$$BEL(\mathbf{C}_i) \leftarrow \eta\, \Phi_i \prod_k M_{ki}^{t}$$

terative Joingraph Propagation

Let $\Pr(\mathbf{X})$ be a distribution induced by a Bayesian network $\mathcal{N}$ having families $X\mathbf{U}$, and let $\mathbf{C}_i$ and $\mathbf{S}_{ij}$ be the clusters and separators of a joingraph for $\mathcal{N}$.
Then messages M_{ij} are a fixed point of IJGP if and only if IJGP marginals $\mu_{\mathbf{c}_i} = BEL(\mathbf{c}_i)$ and $\mu_{\mathbf{s}_{ij}} = BEL(\mathbf{s}_{ij})$ are a stationary point of:

$$\begin{aligned} & \mathrm{ENT}'(\mathbf{X}|\mathbf{e}) + \sum_{\mathbf{C}_i} \mathrm{AVG}'(\log \Phi_i) \\ = \quad & -\sum_{\mathbf{C}_i}\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} \log \mu_{\mathbf{c}_i} + \sum_{\mathbf{S}_{ij}}\sum_{\mathbf{s}_{ij}} \mu_{\mathbf{s}_{ij}} \log \mu_{\mathbf{s}_{ij}} + \sum_{\mathbf{C}_i}\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} \log \Phi_i(\mathbf{c}_i), \end{aligned}$$

under normalization constraints:

$$\sum_{\mathbf{c}_i} \mu_{\mathbf{c}_i} = \sum_{\mathbf{s}_{ij}} \mu_{\mathbf{s}_{ij}} = 1$$

for each cluster $\mathbf{C}_i$ and separator $\mathbf{S}_{ij}$, and under consistency constraints:

$$\sum_{\mathbf{c}_i \sim \mathbf{s}_{ij}} \mu_{\mathbf{c}_i} = \mu_{\mathbf{s}_{ij}} = \sum_{\mathbf{c}_j \sim \mathbf{s}_{ij}} \mu_{\mathbf{c}_j}$$

for each separator $\mathbf{S}_{ij}$ and neighboring clusters $\mathbf{C}_i$ and $\mathbf{C}_j$.

Summary of IJGP so far

A spectrum of approximations.

IBP: results from applying IJGP to the dual joingraph.

Jointree algorithm: results from applying IJGP to a jointree (as a joingraph).

In between these two ends, we have a spectrum of joingraphs and corresponding factorizations, where IJGP seeks stationary points of the KL–divergence between these factorizations and the original distribution.