Chapter 5: General search strategies: Look-ahead

ICS 275

Fall 2010

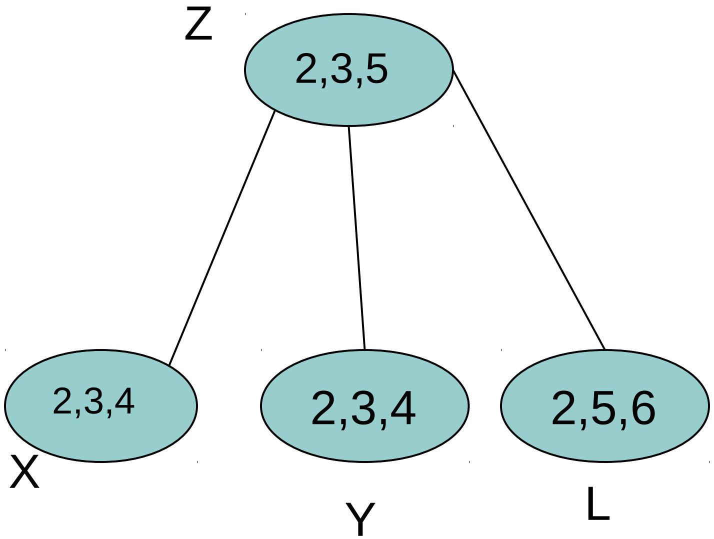
What if the Constraint network is not backtrack-free?

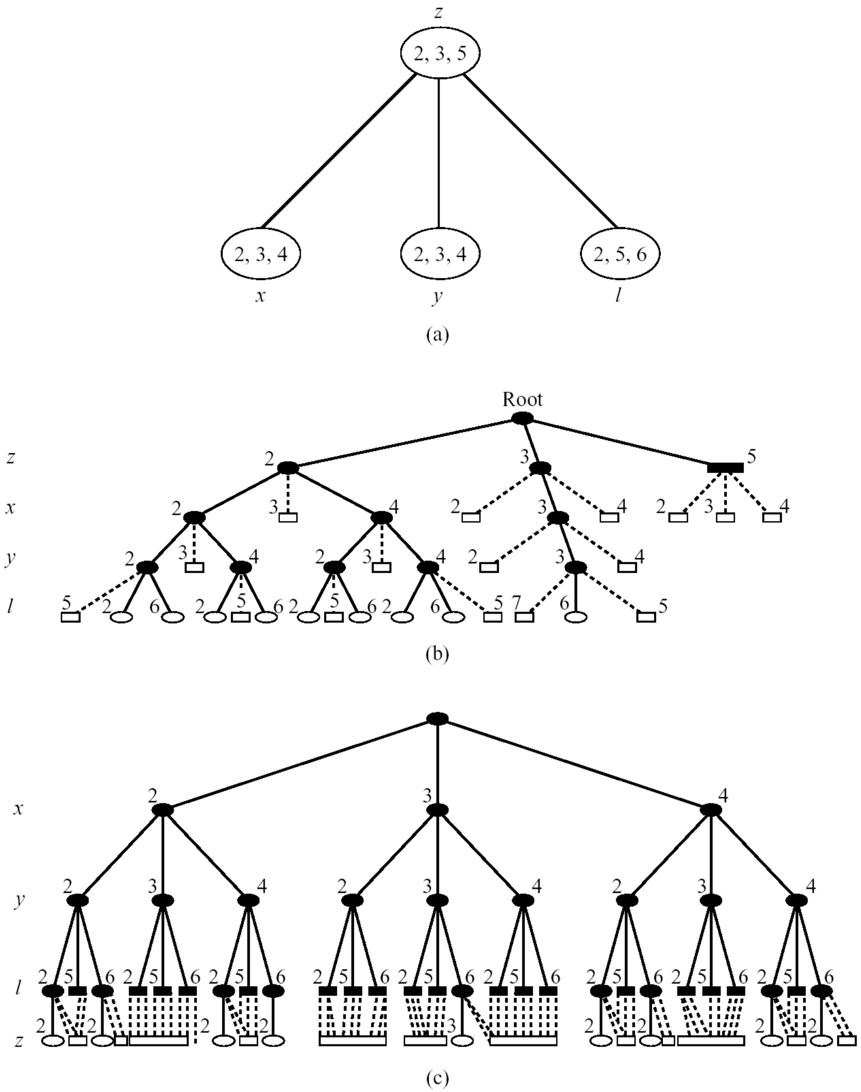
- Backtrack-free in general is too costly so what to do?
- Search?
- What is the search space?
- How to search it? Breadth-first? Depth-first?

The search space for a CN

- A tree of all partial solutions
- A partial solution: (a1,…,aj) satisfying all relevant constraints
- The size of the underlying search space depends on:
  - Variable ordering
  - Level of consistency possessed by the problem

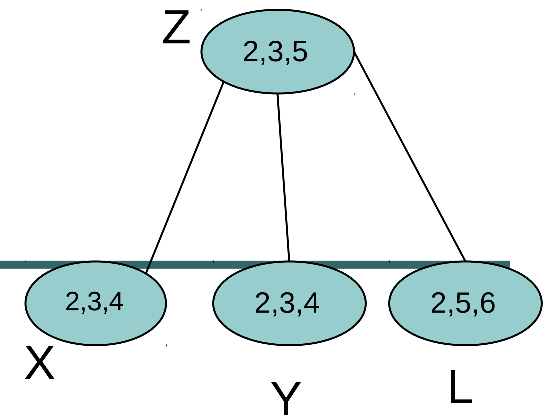
Search space and the effect of ordering

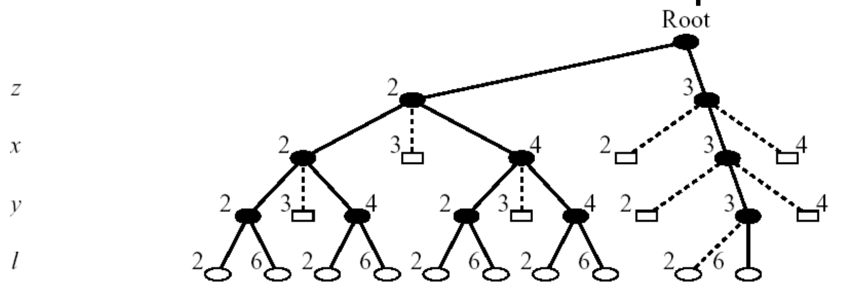




Dependency on consistency level

- After arc-consistency z=5 and l=5 are removed
- After path-consistency
  - R'_zx
  - R'_zy
  - R'_zl
  - R'_xy
  - R'_xl
  - R'_yl





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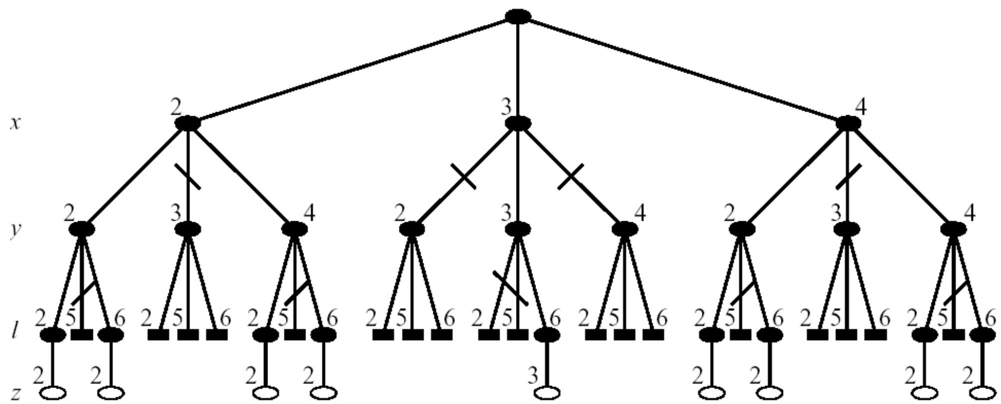


(b)

The effect of higher consistency on search

Theorem 5.1.3 *Let $\mathcal{R}'$ be a tighter network than $\mathcal{R}$, where both represent the same set of solutions. For any ordering d, any path appearing in the search graph derived from $\mathcal{R}'$ also appears in the search graph derived from $\mathcal{R}$.* $\square$

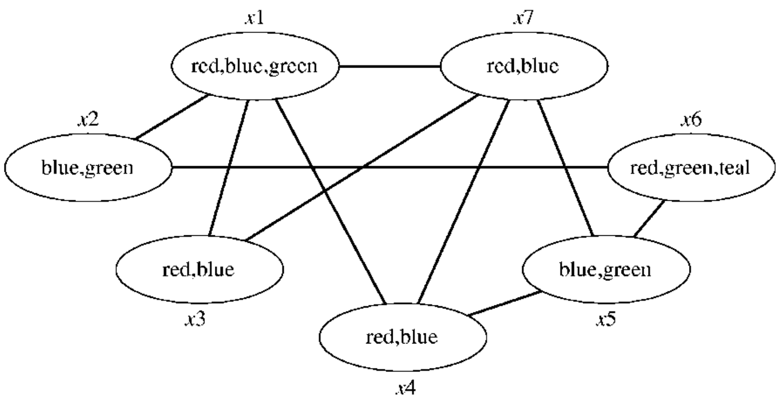
Cost of node's expansion

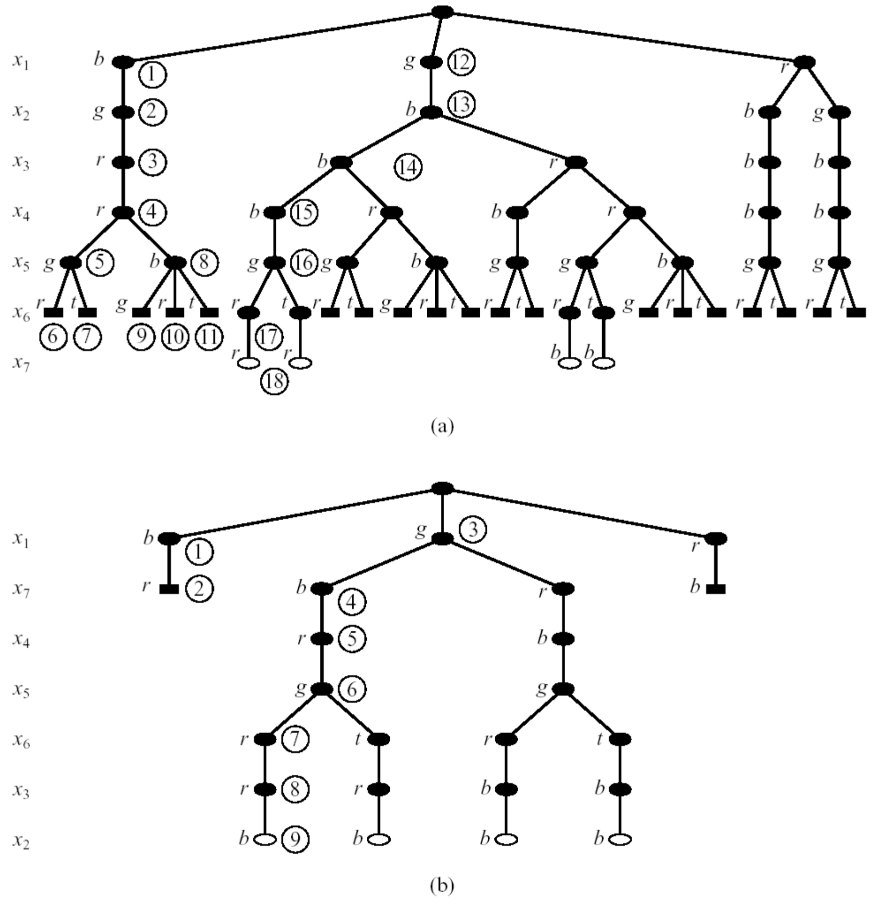
- Number of consistency checks for toy problem:
  - For d1: 19 for R, 43 for R'
  - For d2: 91 on R and 56 on R'

- Reminder:

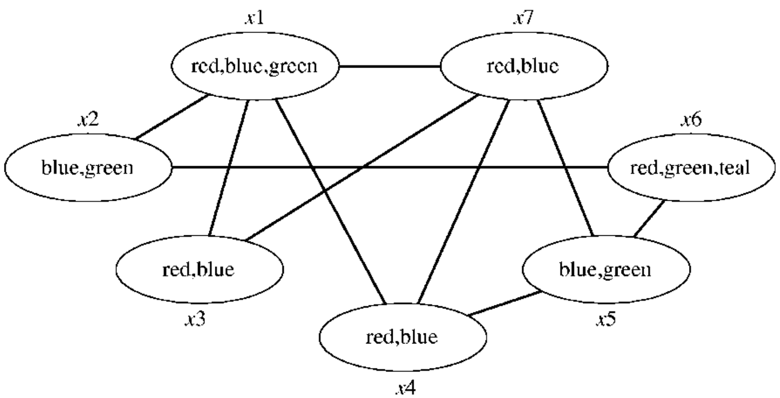
Definition 5.1.5 (backtrack-free network) *A network R is said to be* backtrack-free *along ordering d if every leaf node in the corresponding search graph is a solution.*

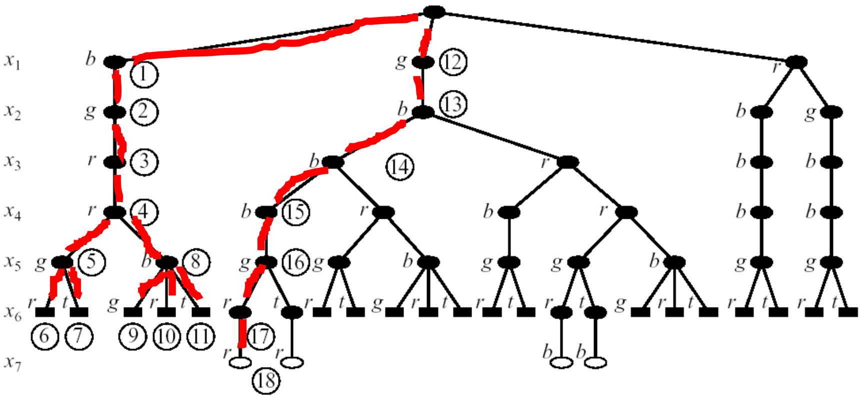
Backtracking Search for a Solution



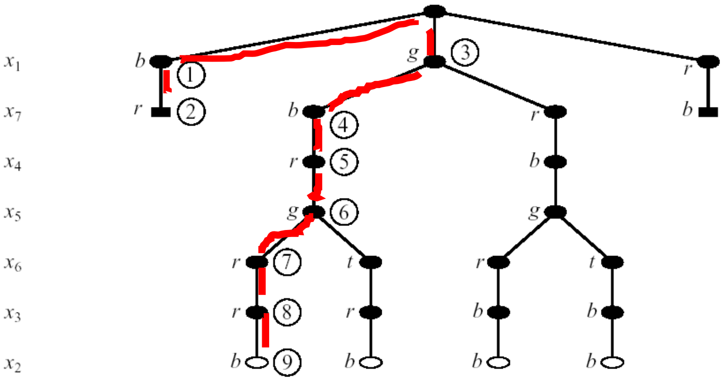


Backtracking Search for a single Solution



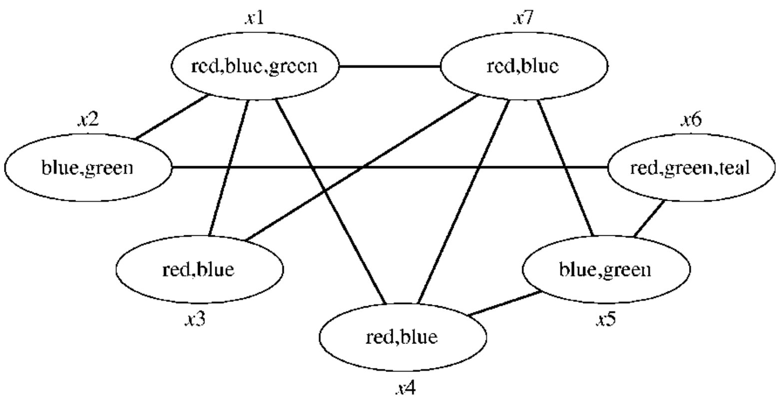


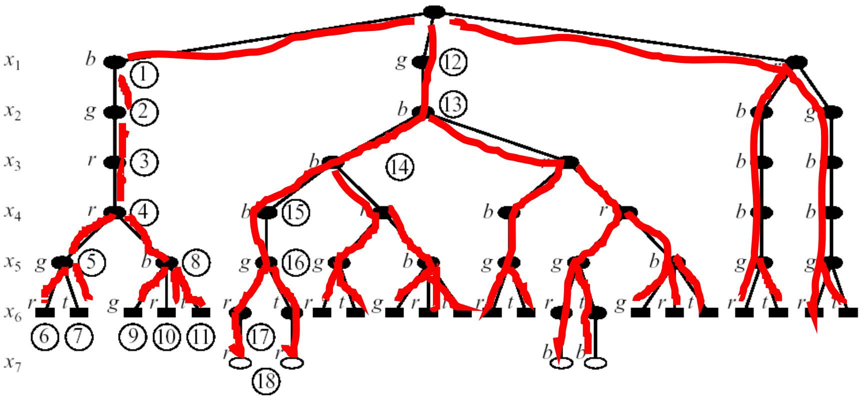
(a)



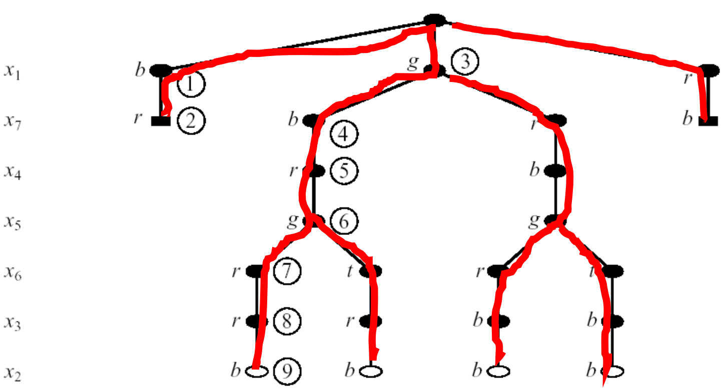
(b)

Backtracking Search for *All* Solutions



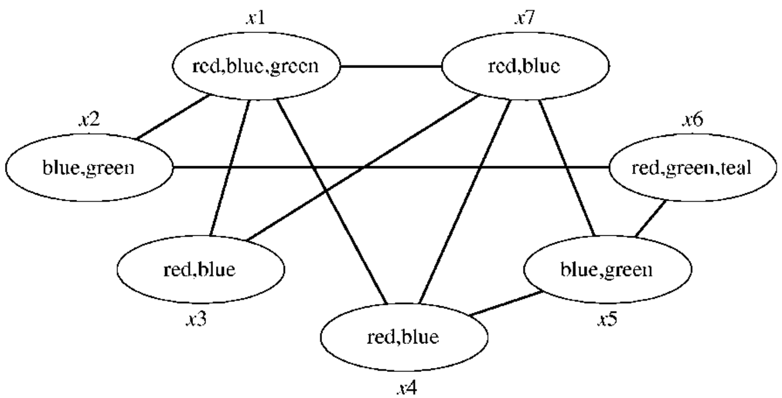


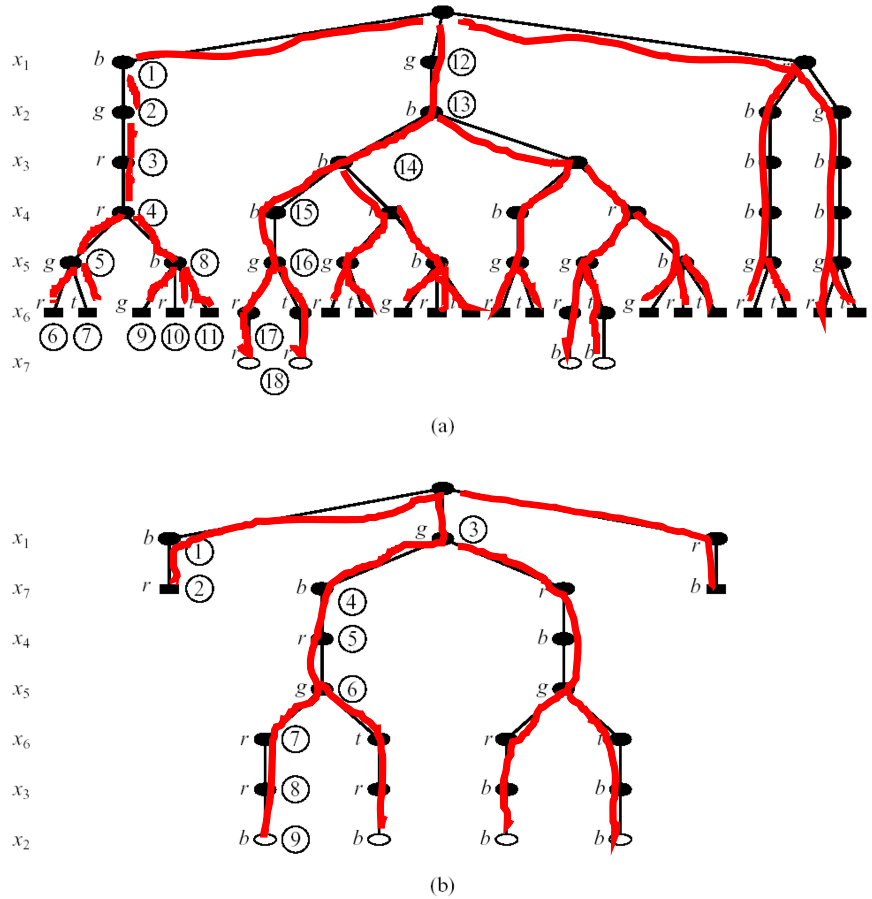
(a)



(b)

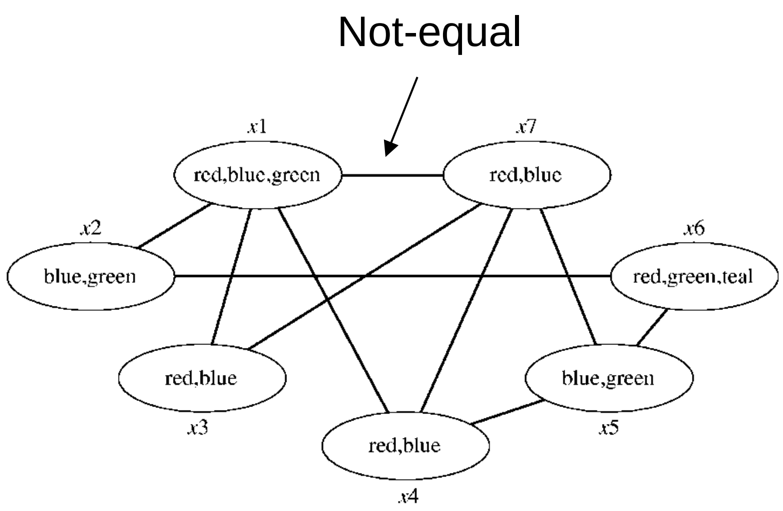
Backtracking Search for *All* Solutions

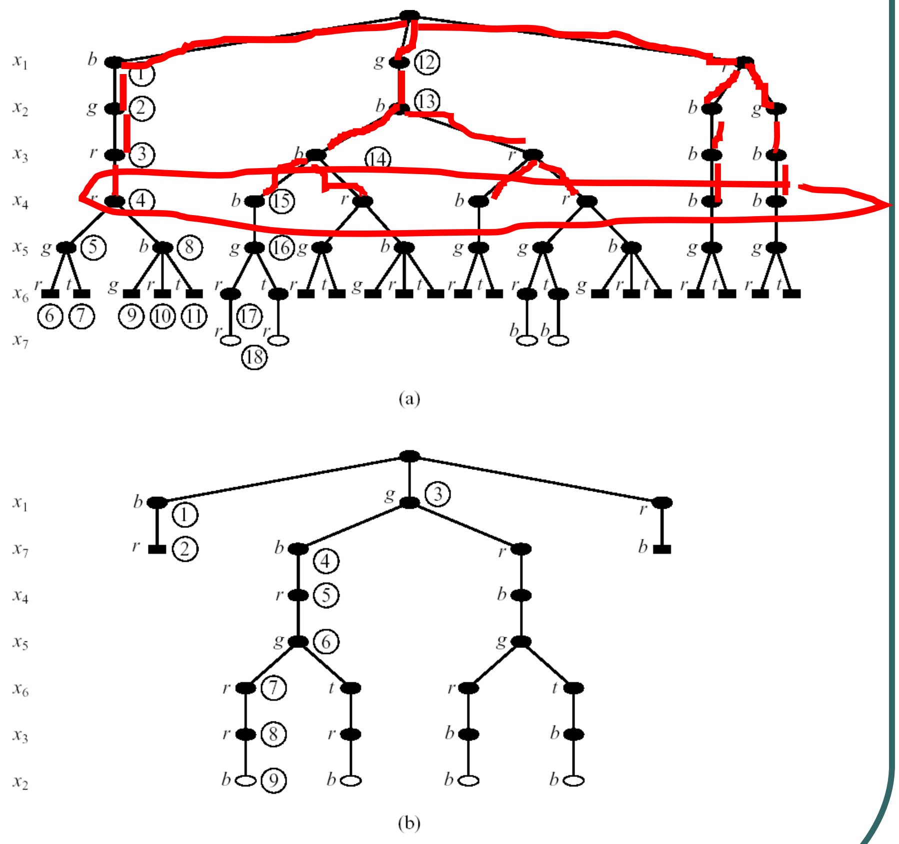




For all tasks
Time: O(exp(n))
Space: linear

Traversing Breadth-First (BFS)?





BFS space is exp(n) while no Time gain → use DFS

Backtracking

```
procedure BACKTRACKING
Input: A constraint network P = (X, D, C).
Output: Either a solution, or notification that the network is inconsistent.

    i ← 1                              (initialize variable counter)
    D'_i ← D_i                         (copy domain)
    while 1 ≤ i ≤ n
        instantiate x_i ← SELECTVALUE
        if x_i is null                 (no value was returned)
            i ← i − 1                  (backtrack)
        else
            i ← i + 1                  (step forward)
            D'_i ← D_i
    end while
    if i = 0
        return "inconsistent"
    else
        return instantiated values of {x_1, ..., x_n}
end procedure

subprocedure SELECTVALUE   (return a value in D'_i consistent with ā_{i−1})

    while D'_i is not empty
        select an arbitrary element a ∈ D'_i, and remove a from D'_i
        if CONSISTENT(ā_{i−1}, x_i = a)
            return a
    end while
    return null                        (no consistent value)
end procedure
```

Figure 5.4: The backtracking algorithm.

- Complexity of extending a partial solution:
  - Complexity of consistent O(e log t), t bounds tuples, e constraints
  - Complexity of selectValue O(e k log t)

Improving backtracking

- Before search: (reducing the search space)
  - Arc-consistency, path-consistency
  - Variable ordering (fixed)
- During search:
  - Look-ahead schemes:
    - value ordering,
    - variable ordering (if not fixed)
  - Look-back schemes:
    - Backjump
    - Constraint recording
    - Dependency-directed backtacking

Look-ahead: value orderings

- Intuition:
  - Choose value least likely to yield a dead-end
  - Approach: apply propagation at each node in the search tree
- Forward-checking
  - (check each unassigned variable separately
- Maintaining arc-consistency (MAC)
  - (apply full arc-consistency)
- Full look-ahead
  - One pass of arc-consistency (AC-1)
- Partial look-ahead
  - directional-arc-consistency

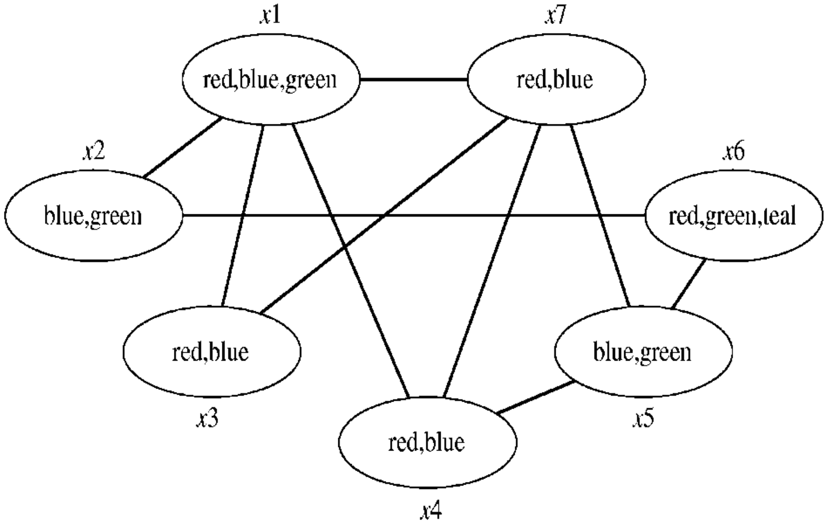
Generalized look-ahead

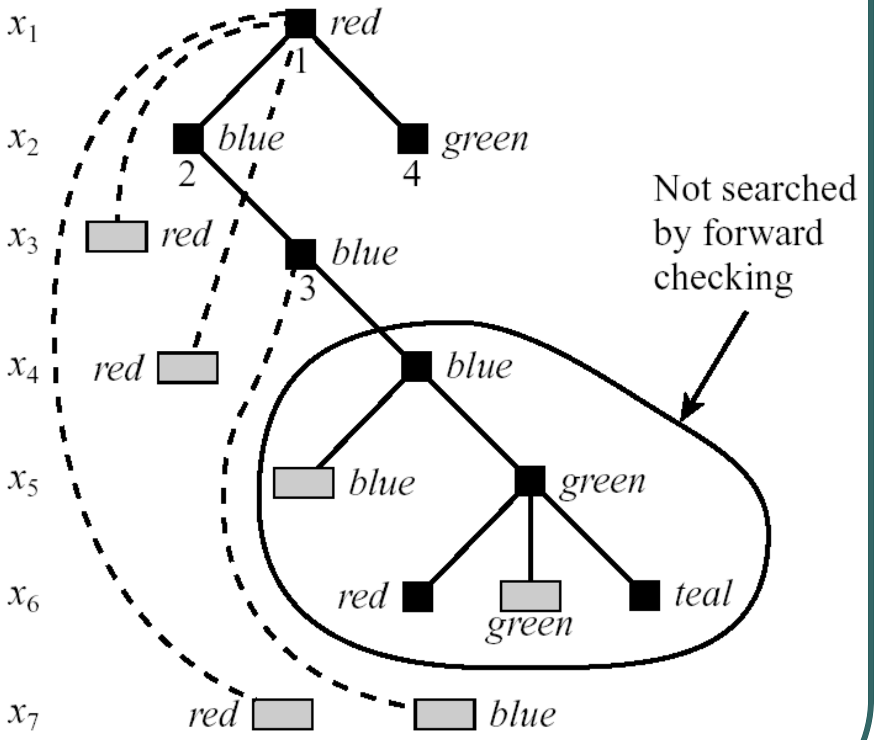
```
procedure GENERALIZED-LOOKAHEAD
Input: A constraint network P = (X, D, C)
Output: Either a solution, or notification that the network is inconsis-
tent.

    D'_i ← D_i for 1 ≤ i ≤ n        (copy all domains)
    i ← 1                           (initialize variable counter)
    while 1 ≤ i ≤ n
       instantiate x_i ← SELECTVALUE-XXX
       if x_i is null               (no value was returned)
          i ← i − 1                 (backtrack)
          reset each D'_k, k > i, to its value before x_i was last instantiated
       else
          i ← i + 1                 (step forward)
    end while
    if i = 0
       return "inconsistent"
    else
       return instantiated values of {x_1, ..., x_n}
end procedure
```

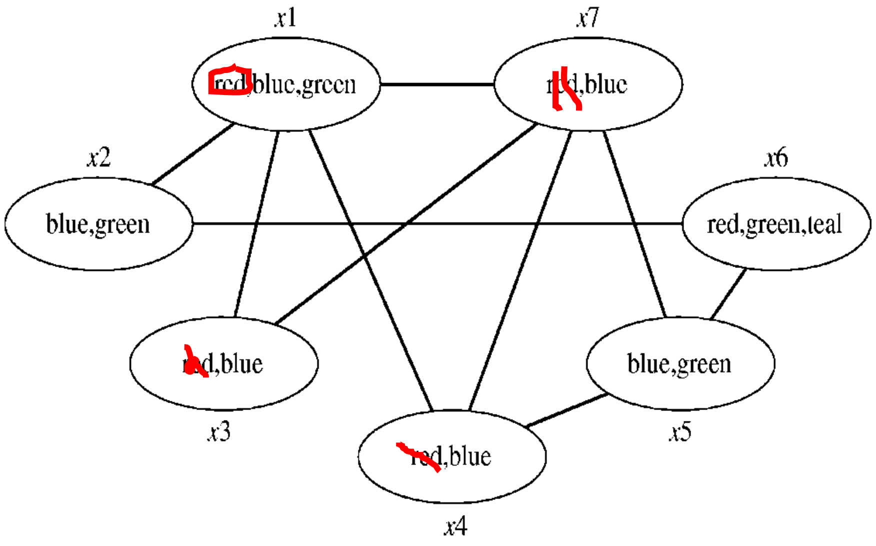
Figure 5.7: A common framework for several look-ahead based search algorithms. By replacing SELECTVALUE-XXX with SELECTVALUE-FORWARD-CHECKING, the forward checking algorithm is obtained. Similarly, using SELECTVALUE-ARC-CONSISTENCY yields an algorithm that interweaves arc-consistency and search.

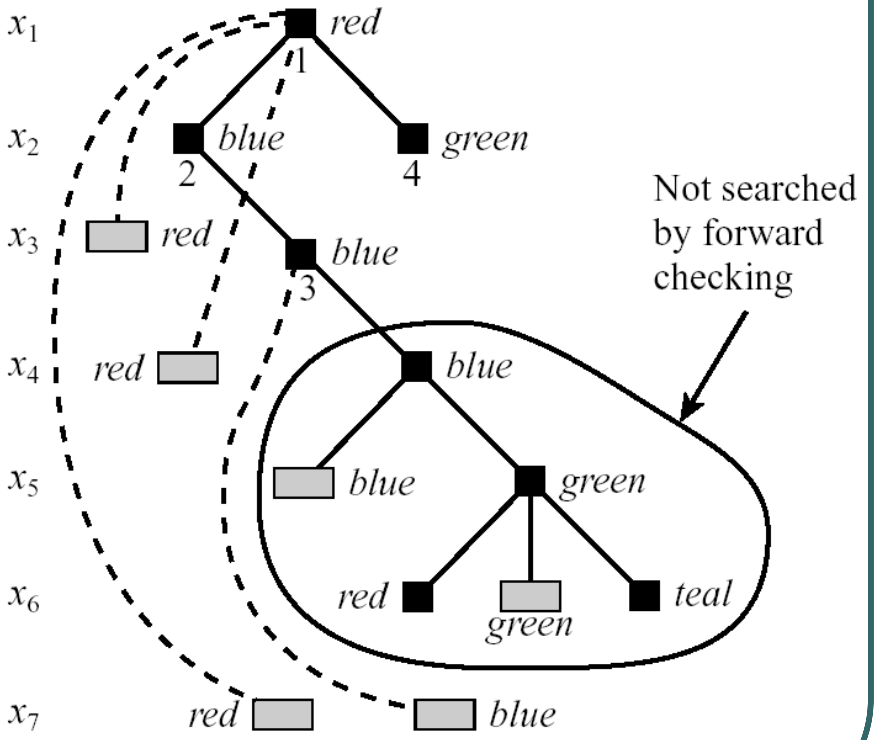
Forward-checking example



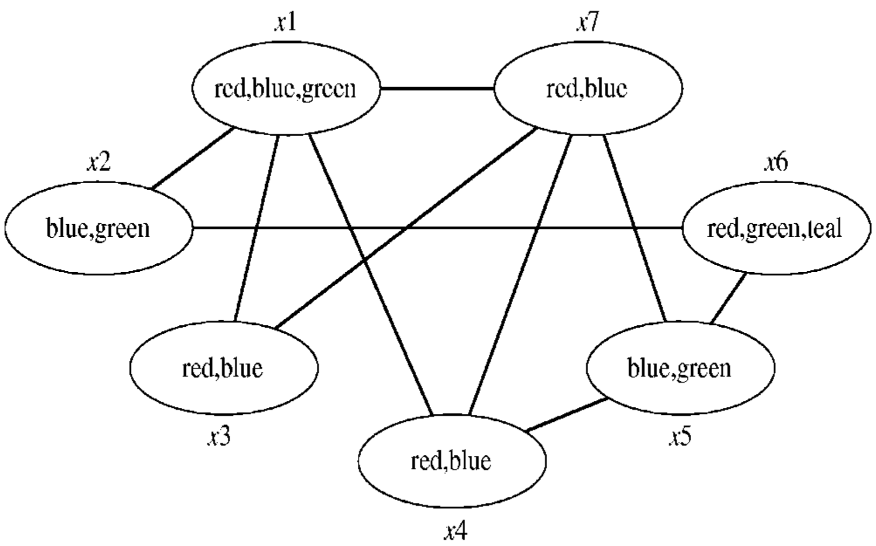


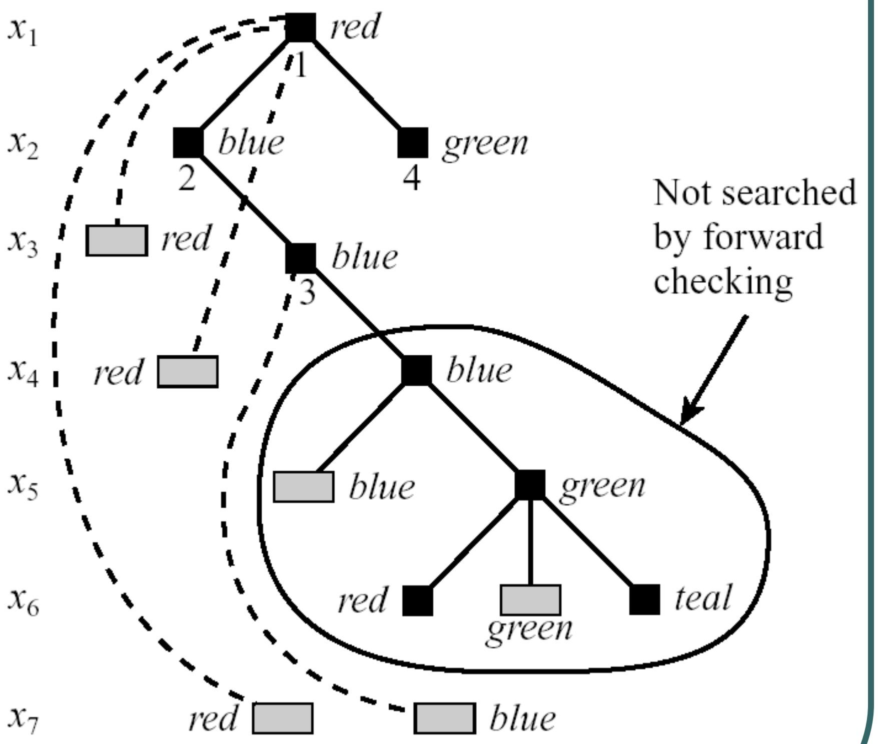
Forward-Checking for Value Selection





Forward-Checking for Value Ordering

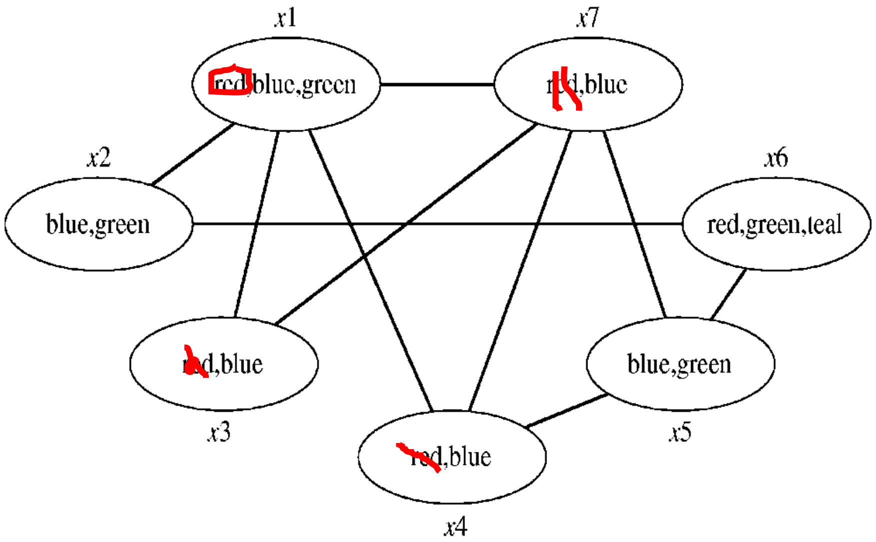




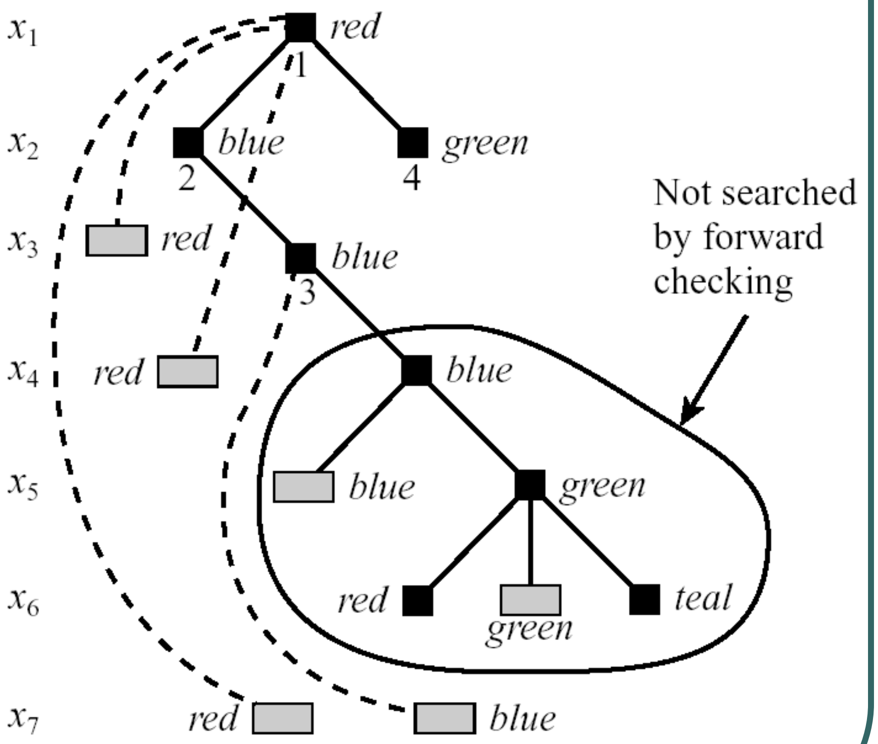
FC overhead: $O(ek^2)$

For each value of a future variable e_u
Tests: O(k e_u), for all future variables O(ke)
For all current domain O(k^2 e)

Forward-Checking for Value Ordering



FW overhead: : $O\left(ek^2\right)$



Forward-checking

```
procedure SELECTVALUE-FORWARD-CHECKING
    while D'_i is not empty
        select an arbitrary element a ∈ D'_i, and remove a from D'_i
        empty-domain ← false
        for all k, i < k ≤ n
            for all values b in D'_k
                if not CONSISTENT(a⃗_{i-1}, x_i = a, x_k = b)
                    remove b from D'_k
            end for
            if D'_k is empty          (x_i = a leads to a dead-end)
                empty-domain ← true
        if empty-domain               (don't select a)
            reset each D'_k, i < k ≤ n to value before a was selected
        else
            return a
    end while
    return null                       (no consistent value)
end procedure
```

Figure 5.8: The SELECTVALUE subprocedure for the forward checking algorithm.

Complexity of selectValue-forward-checking at each node: $O(ek^2)$

Arc-consistency look-ahead
(Gashnig, 1977)

- Applies full arc-consistency on all un-instantiated variables following each value assignment to the current variable.
- Complexity:
  - If optimal arc-consistency is used: $O\left(ek^3\right)$
  - What is the complexity overhead when AC-1 is used at each node?

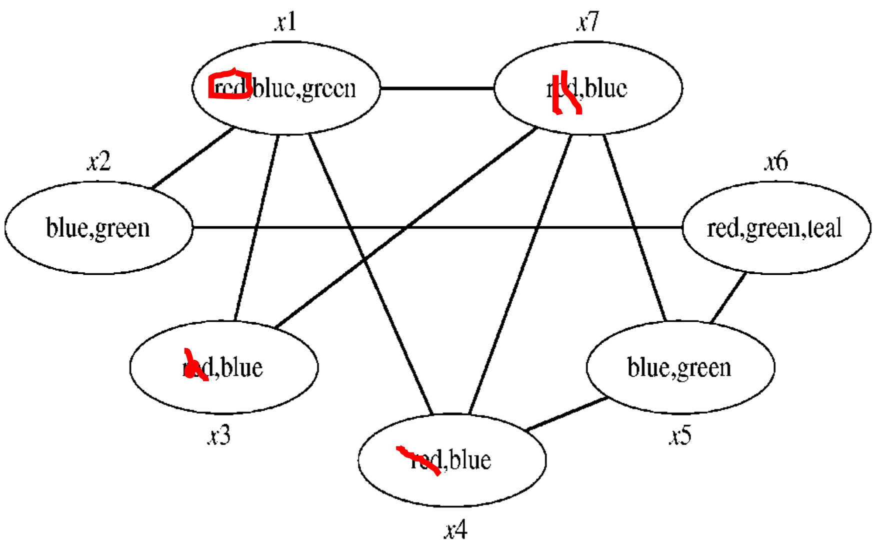
Forward-checking: $O\left(ek^2\right)$

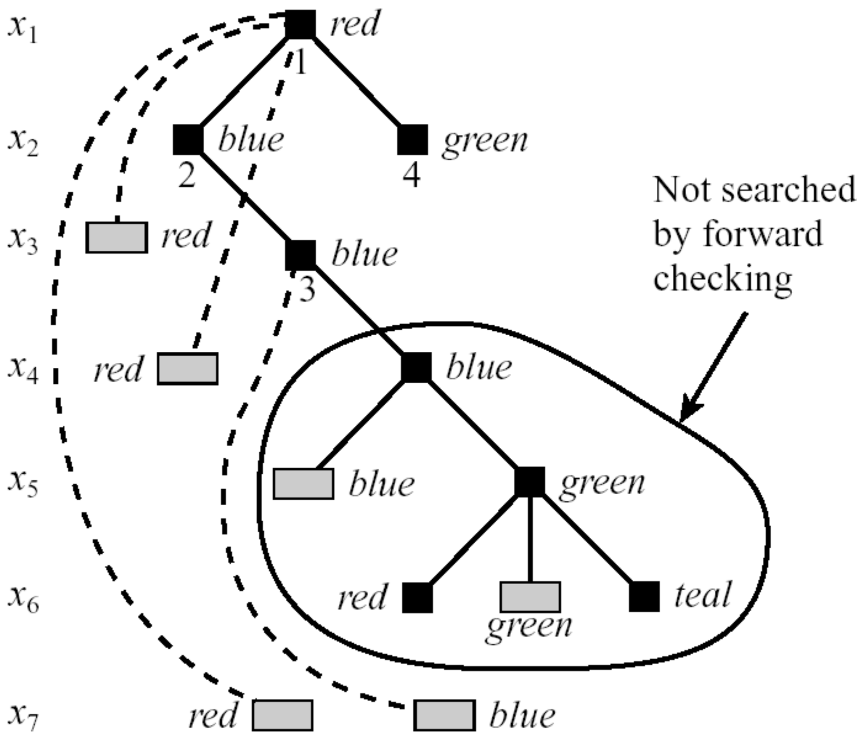
MAC: $O\left(ek^3\right)$

MAC: Mmaintaining arc-consistency (Sabin and Freuder 1994)

- Perform arc-consistency in a binary search tree: Given a domain X={1,2,3,4} the algorithm assigns X=1 (and apply arc-consistency) and if x=1 is pruned, it applies arc-consistency to X={2,3,4}
- If inconsistency is discovered, a new variable is selected (not necessarily X)

MAC for Value Ordering



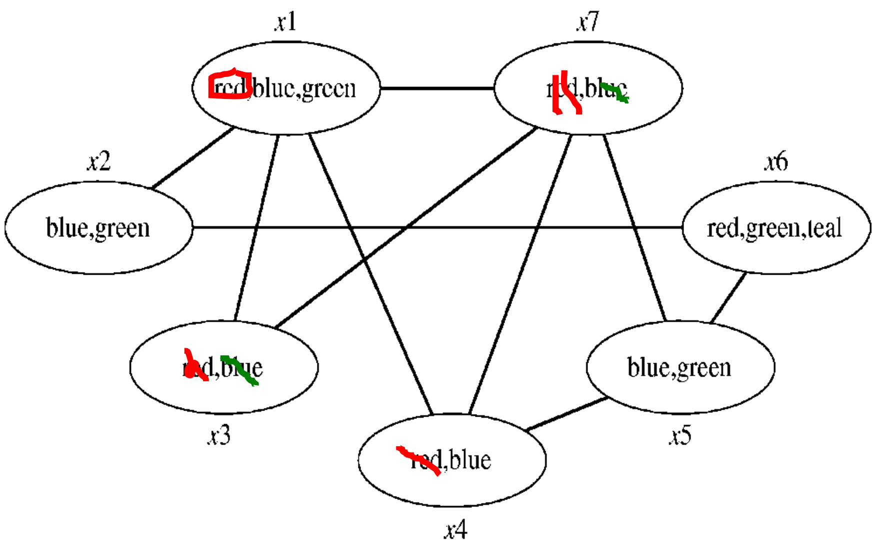


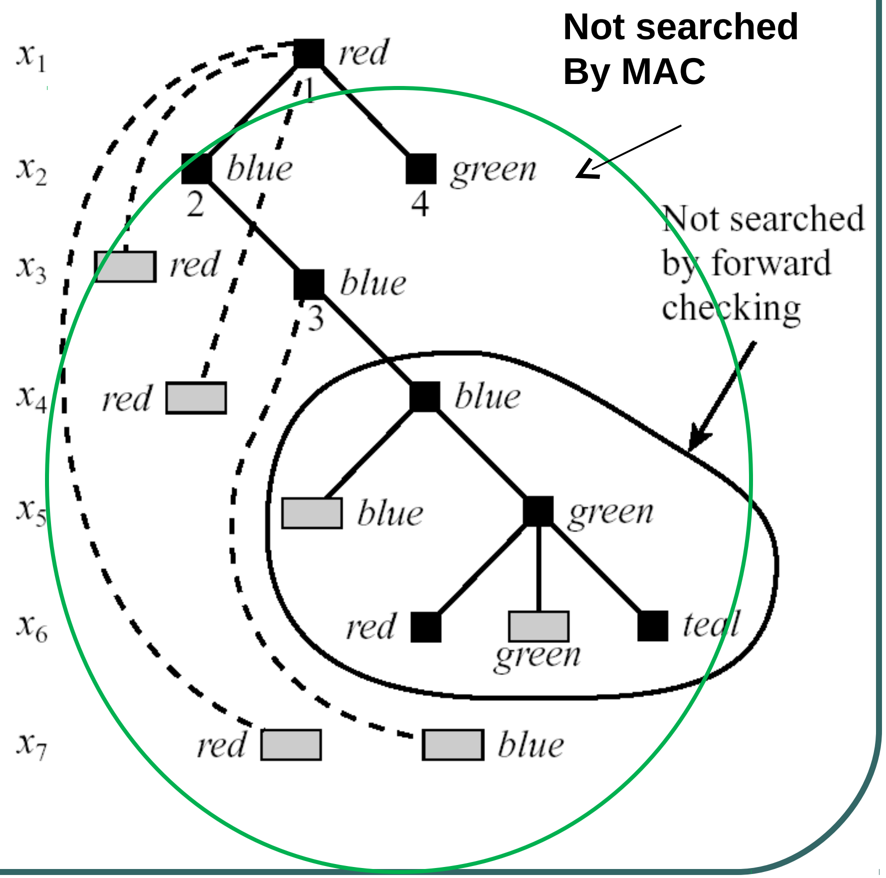
FW overhead: $O(ek^2)$

MAC overhead: $O(ek^3)$

MAC for Value Ordering

Arc-consistency prunes x1=red
Prunes the whole tree





FW overhead: $O(ek^2)$

MAC overhead: $O(ek^3)$

Arc-consistency look-ahead:
(a variant: maintaining arc-consistency MAC)

```
subprocedure SELECTVALUE-ARC-CONSISTENCY

    while D'_i is not empty
       select an arbitrary element a ∈ D'_i, and remove a from D'_i
       repeat
       removed-value ← false
          for all j, i < j ≤ n
             for all k, i < k ≤ n
                for each value b in D'_j
                   if there is no value c ∈ D'_k such that
                         CONSISTENT(ā_{i-1}, x_i = a, x_j = b, x_k = c)
                      remove b from D'_j
                      removed-value ← true
                end for
             end for
          end for
       until removed-value = false
       if any future domain is empty    (don't select a)
          reset each D'_j, i < j ≤ n, to value before a was selected
       else
          return a
    end while
    return null                    (no consistent value)
end procedure
```

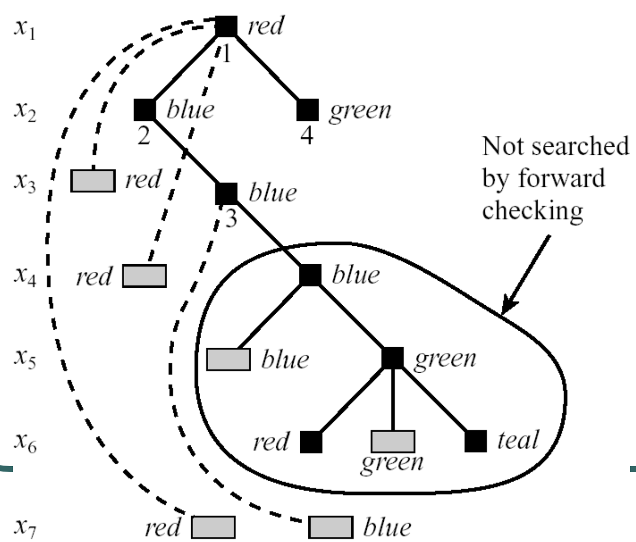
Figure 5.10: The SELECTVALUE subprocedure for arc-consistency, based on the AC-1 algorithm.

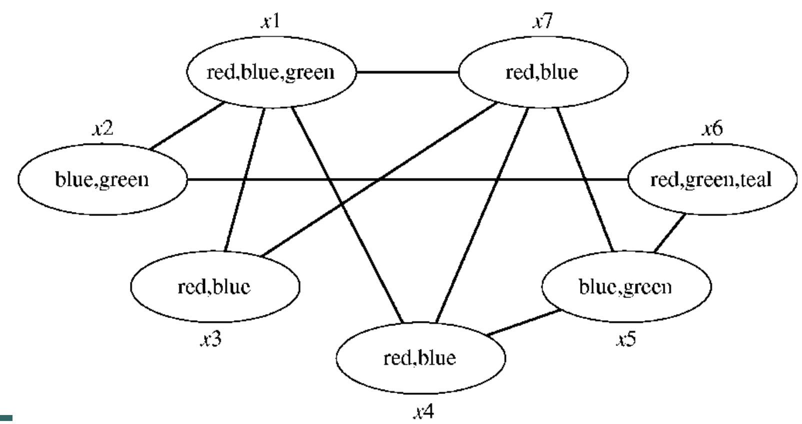
Full and partial look-ahead

- Full looking ahead:
  - Make one pass through future variables (delete, repeat-until)
- Partial look-ahead:
  - Applies (similar-to) directional arc-consistency to future variables.
  - Complexity: also $O\left(ek^3\right)$
  - More efficient than MAC

Example of partial look-ahead

Example 5.3.3 Conside the problem in Figure 5.3 using the same ordering of variables and values as in Figure 5.9. Partial-look-ahead starts by considering $x_1 = red$. Applying directional arc-consistency from x_1 towards x_7 will first shrink the domains of x_3, x_4 and x_7, (when processing x_1), as was the case for forward-checking. Later, when directional arc-consistency processes x_4 (with its only value, "blue") against x_7 (with its only value, "blue"), the domain of x_4 will become empty, and the value "red" for x_1 will be rejected. Likewise, the value $x_1 = blue$ will be rejected. Therefore, the whole tree in Figure 5.9 will not be visited if either partial-look-ahead or the more extensive look-ahead schemes are used. With this level of look-ahead only the subtree below $x_1 = green$ will be expanded.
□





Branching-ahead: Dynamic Value Ordering

Rank order the promise in non-rejected values

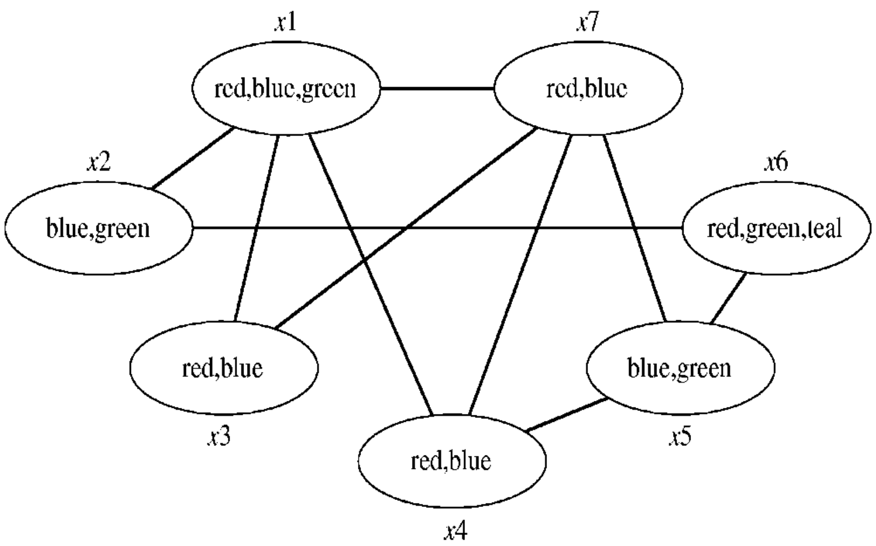
- Rank functions
  - MC (min conflict)
  - MD (min domain)
  - SC (expected solution counts)

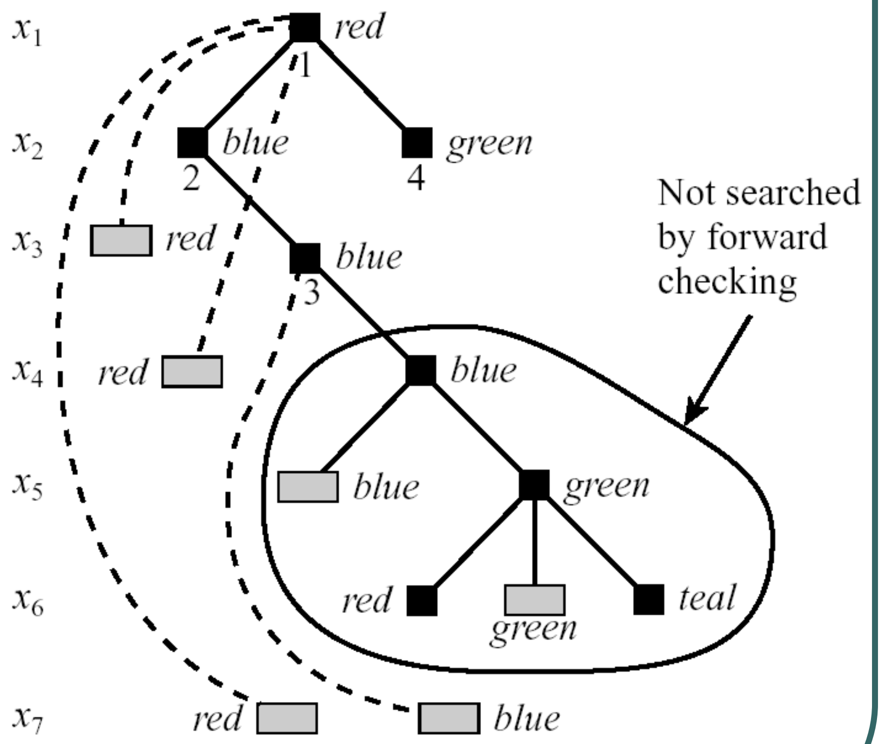
- MC results (Frost and Dechter, 1996)
- SC – currently shows good performance using IJGP (Kask, Dechter and Gogate, 2004)

Dynamic Variable Ordering (DVO)

- Following constraint propagation, choose the most constrained variable
- **Intuition:** early discovery of dead-ends
- **Highly effective**: the single most important heuristic to cut down search space
- Most popular with FC
- Dynamic search rearrangement (Bitner and Reingold, 1975) (Purdon,1983)

Forward-Checking, Variable Ordering



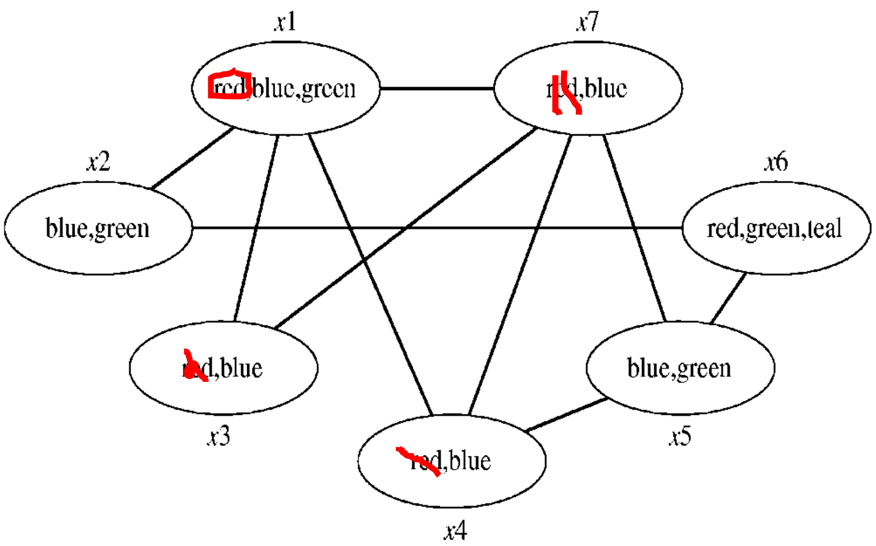


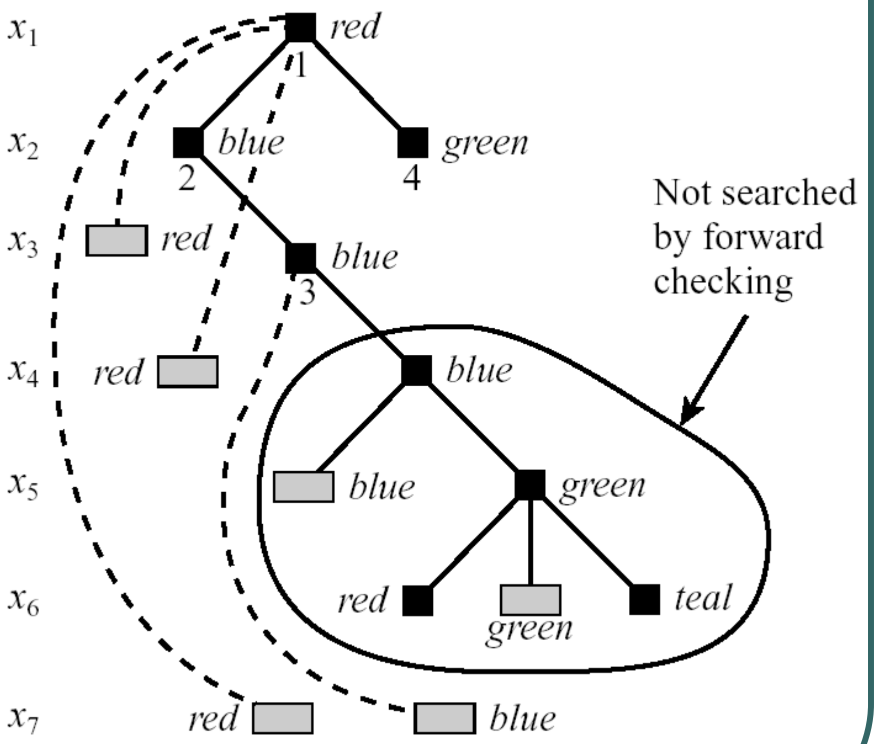
FW overhead: $O\left(ek^2\right)$

MAC overhead: $O\left(ek^3\right)$

Forward-Checking, Variable Ordering

After X1 = red choose X3 and not X2



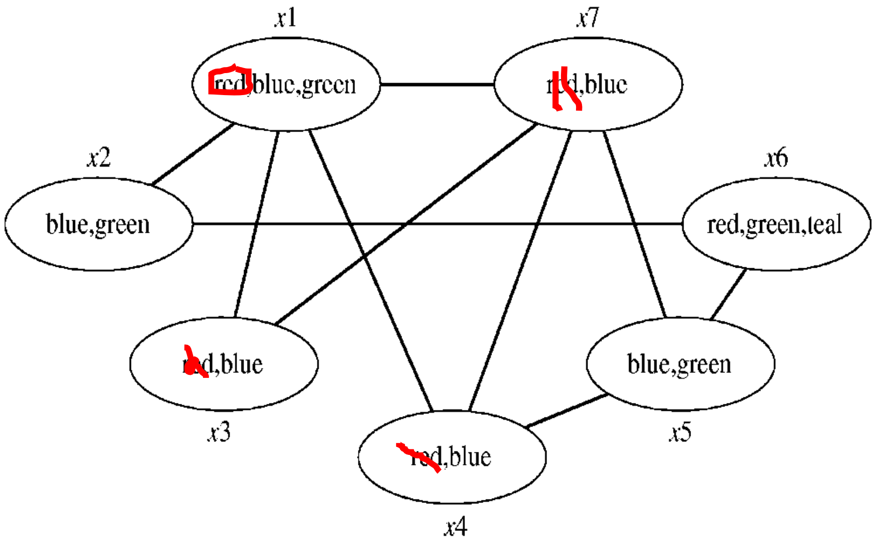


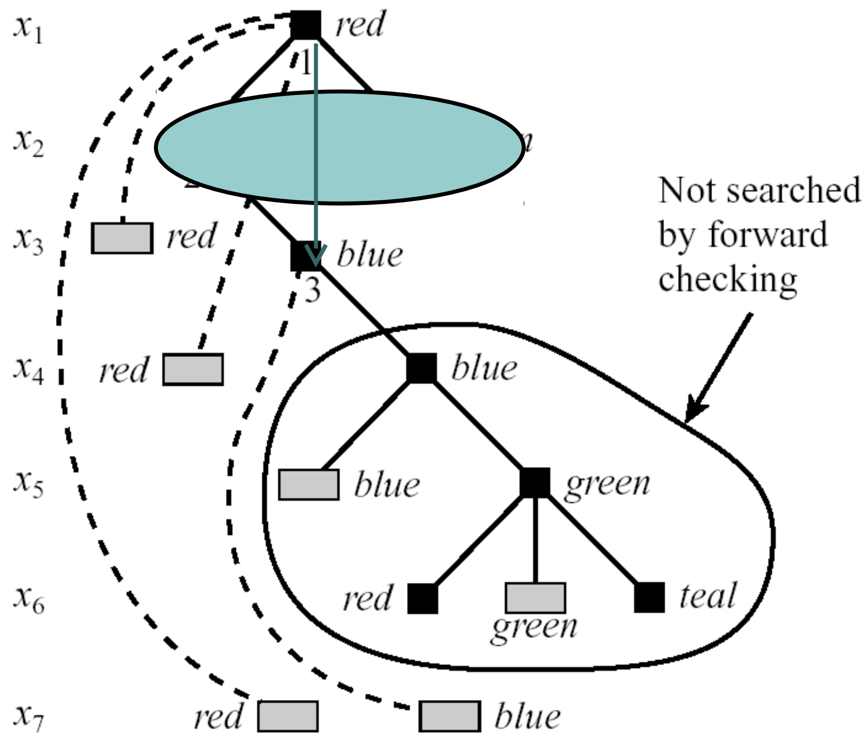
FW overhead: $O(ek^2)$

MAC overhead: $O(ek^3)$

Forward-Checking, Variable Ordering

After X1 = red choose X3 and not X2



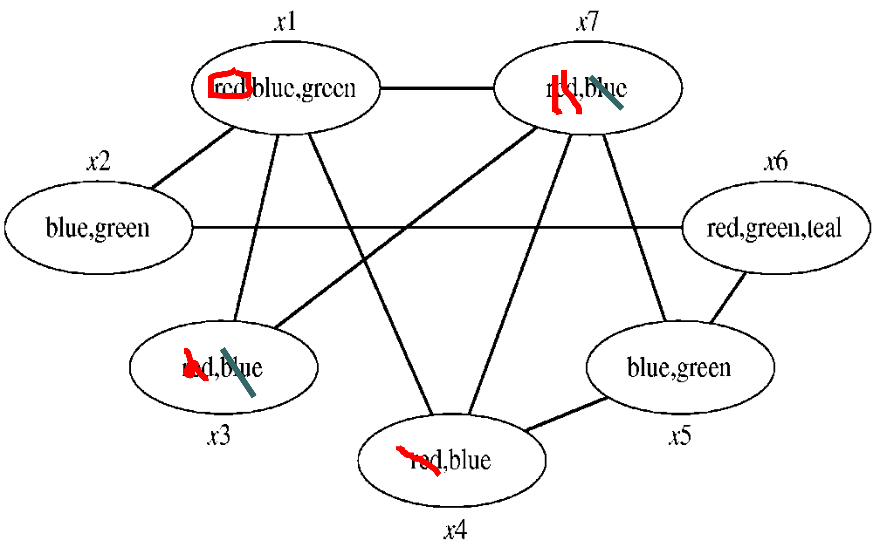


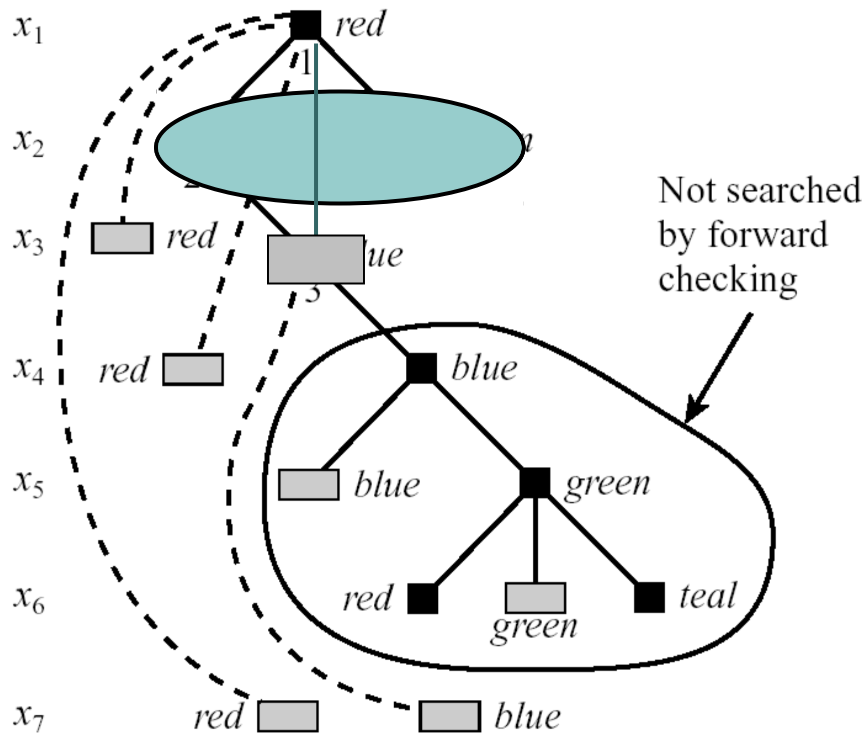
FW overhead: $O\left(ek^2\right)$

MAC overhead: $O\left(ek^3\right)$

Forward-Checking, Variable Ordering

After X1 = red choose X3 and not X2

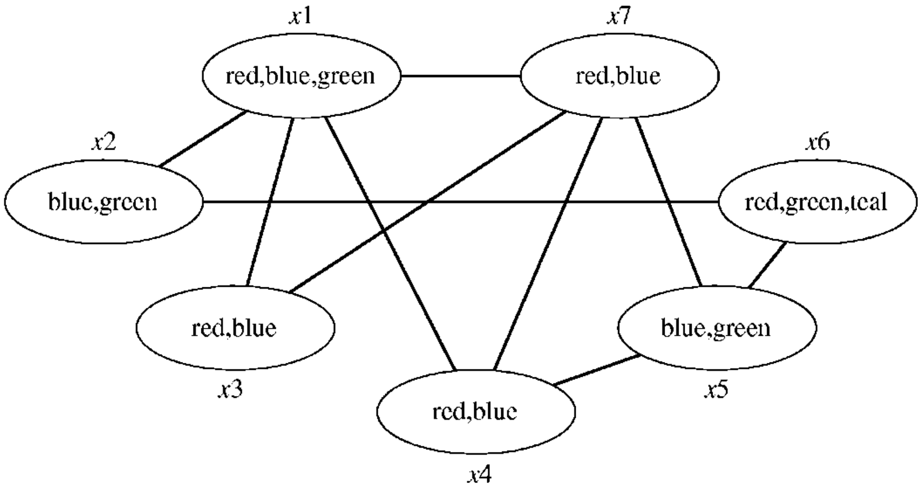




FW overhead: $O\left(ek^2\right)$

MAC overhead: $O\left(ek^3\right)$

Example: DVO with forward checking (DVFC)



Example 5.3.4 Consider again the example in Figure 5.3. Initially, all variables have domain size of 2 or more. DVFC picks x_7, whose domain size is 2, and the value $< x_7, blue >$. Forward-checking propagation of this choice to each future variable restricts the domains of x_3, x_4 and x_5 to single values, and reduces the size of x_1's domain by one. DVFC selects x_3 and assigns it its only possible value, *red*. Subsequently, forward-checking causes variable x_1 to also have a singleton domain. The algorithm chooses x_1 and its only consistent value, *green*. After propagating this choice, we see that x_4 has one value, *red*; it is selected and assigned the value. Then x_2 can be selected and assigned its only consistent value, *blue*. Propagating this assignment does not further shrink any future domain. Next, x_5 can be selected and assigned *green*. The solution is then completed, without dead-ends, by assigning *red* or *teal* to x_6. □

Algorithm DVO (DVFC)

procedure DVFC
Input: A constraint network $\mathcal{R} = (X, D, C)$
Output: Either a solution, or notification that the network is inconsistent.

$D'_i \leftarrow D_i$ for $1 \leq i \leq n$ (copy all domains)
$i \leftarrow 1$ (initialize variable counter)
$s = \min_{i<j\leq n} |D'_j|$ (find future var with smallest domain)
$x_{i+1} \leftarrow x_s$ (rearrange variables so that x_s follows x_i)
while $1 \leq i \leq n$
instantiate $x_i \leftarrow$ SELECTVALUE-FORWARD-CHECKING
if x_i is null (no value was returned)
reset each D' set to its value before x_i was last instantiated
$i \leftarrow i - 1$ (backtrack)
else
if $i < n$
$i \leftarrow i + 1$ (step forward to x_s)
$s = \min_{i<j\leq n} |D'_j|$ (find future var with smallest domain)
$x_{i+1} \leftarrow x_s$ (rearrange variables so that x_s follows x_i)
$i \leftarrow i + 1$ (step forward to x_s)
end while
if $i = 0$
return "inconsistent"
else
return instantiated values of $\{x_1, \ldots, x_n\}$
end procedure

Figure 5.12: The DVFC algorithm. It uses the SELECTVALUE-FORWARD-CHECKING subprocedure given in Fig. 5.8.

DVO: Dynamic Variable Ordering, More involved heuristics

- *dom*: choose a variable with min domain
- *deg*: choose variable with max degree
- *dom+deg*: dom and break ties with max degree
- *dom/deg* (Bessiere and Ragin, 96): choose min dom/deg
- *dom/wdeg*: domain divided by weighted degree. Constraints are weighted as they get involved in more conflicts. wdeg: sum the weights of all constraints that touch x.

Implementing look-aheads

- Cost of node generation should be reduced
- Solution: keep a table of viable domains for each variable and each level in the tree.

- Space complexity $O(n^2 k)$
- Node generation = table updating $O(e_d k) \Rightarrow O(ek)$

Branching Strategies (selecting the search space)

(see vanBeek, chapter 4 in Handbook)

- Enumeration branching: the naïve backtracking search choice
- A branching strategy in the search tree: a set of branching constraints p(b_1,…b_j} where b_i is a branching constraint
- Branches are often ordered using a heuristic.
- To ensure completeness, the constraints that are ordered on the branches should be exclusive and exhaustive.
- Most common are unary constraints:
  - Enumeration: (x=1,x=2,x=3…)
  - Binary choices: (x=1, x != 1)
  - Domain spliting: (x>3,x<3)
- Using domain-specific formulas
  - Scheduling: one job before or after: (x_1 +d_1 < x_2, x_2+d_2 < x_1)
  - Can be simulated by auxiliary variables.
  - Searching the dual problem
  - Formula-based splitting in SAT

Randomization

- Randomized variable selection (for tie breaking rule)
- Randomized value selection (for tie breaking rule)
- Random restarts with increasing time-cutoff
- Capitalizing on huge performance variance
- All modern SAT solvers that are competitive us restarts.

The cycle-cutset effect

- A cycle-cutset is a subset of nodes in an undirected graph whose removal results in a graph with no cycles
- A constraint problem whose graph has a cycle-cutset of size c can be solved by partial look-ahead in time $O((n-c)k^{(c+2)})$

Extension to stronger look-ahead

- Extend to path-consistency or i-consistency or generalized-arc-consistency

Definition 5.3.7 (general arc-consistency) *Given a constraint* $C = (R, S)$ *and a variable* $x \in S$*, a value* $a \in D_x$ *is supported in* C *if there is a tuple* $t \in R$ *such that* $t[x] = a$*.* t *is then called a support for* $< x, a >$ *in* C*.* C *is arc-consistent if for each variable* x*, in its scope and each of its values,* $a \in D_x$*,* $< x, a >$ *has a support in* C*. A CSP is arc-consistent if each of its constraints is arc-consistent.*

Look-ahead for SAT: DPLL

(Davis-Putnam, Logeman and Laveland, 1962)

DPLL(φ)
Input: A cnf theory φ
Output: A decision of whether φ is satisfiable.
1. Unit_propagate(φ);
2. If the empty clause is generated, return(*false*);
3. Else, if all variables are assigned, return(*true*);
4. Else
5. Q = some unassigned variable;
6. return(**DPLL**($\varphi \wedge Q$) $\vee$ **DPLL**($\varphi \wedge \neg Q$))

Figure 5.13: The DPLL Procedure

What is SAT?

Given a sentence:

- ***Sentence***: conjunction of clauses

$$\left(c_1 \vee \neg c_4 \vee c_5 \vee c_6\right) \wedge \left(c_2 \vee \neg c_3\right) \wedge \left(\neg c_4\right)$$

- ***Clause***: disjunction of literals

$$\left(c_2 \vee \neg c_3\right)$$

- ***Literal***: a term or its negation

$$c_1, \neg c_6$$

- ***Term***: Boolean variable

$$c_1 = 1 \Leftrightarrow \neg c_1 = 0$$

Question: Find an assignment of truth values to the Boolean variables such the sentence is satisfied.

CSP is NP-Complete

- Verifying that an assignment for all variables is a solution
  - Provided constraints can be checked in polynomial time
- Reduction from 3SAT to CSP
  - Many such reductions exist in the literature (perhaps 7 of them)

Problem reduction

Example: CSP into SAT (*proves nothing, just an exercise*)

Notation: variable-value pair = **vvp**

- vvp $\rightarrow$ term
  - V_1 = {a, b, c, d} yields $x_1 = (V_1, a)$, $x_2 = (V_1, b)$, $x_3 = (V_1, c)$, $x_4 = (V_1, d)$,
  - V_2 = {a, b, c} yields $x_5 = (V_2, a)$, $x_6 = (V_2, b)$, $x_7 = (V_2, c)$.
- The vvp's of a variable $\rightarrow$ disjunction of terms $x_1 \vee x_2 \vee x_3 \vee x_4$
  - V_1 = {a, b, c, d} yields
- (Optional) At most one VVP per variable

$$(x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4) \vee ¿$$

$$¿ (\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4)$$

CSP into SAT (cont.)

Constraint: $C_{V_1 V_2} = \{(a,a),(a,b),(b,c),(c,b),(d,a)\}$

1. Way 1: Each inconsistent tuple $\rightarrow$ one disjunctive clause
   - For example: $\neg x_1 \vee \neg x_7$ how many?

1. Way 2:
   a) Consistent tuple $\rightarrow$ conjunction of terms $x_1 \wedge x_5$
   b) Each constraint $\rightarrow$ disjunction of these conjunctions

$$(x_1 \wedge x_5) \vee (x_1 \wedge x_6) \vee (x_2 \wedge x_7)$$
$$¿(x_3 \wedge x_6) \vee (x_4 \wedge x_5)$$

$\rightarrow$ transform into conjunctive normal form (CNF)

Question: find a truth assignment of the Boolean variables such that the sentence is satisfied

Example of DPLL

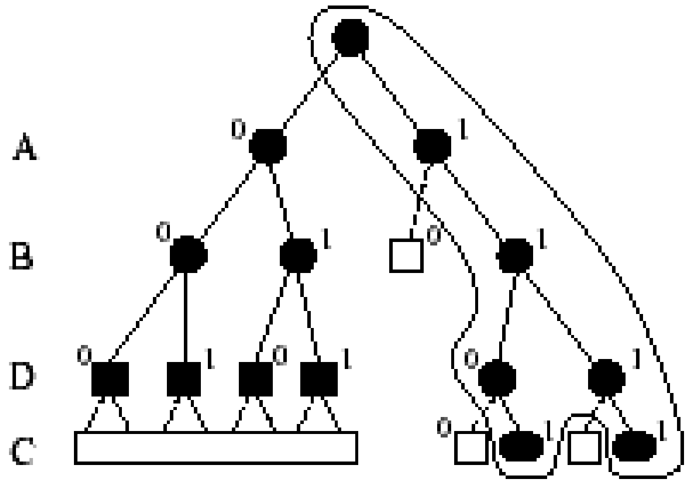


Figure 5.14: A backtracking search tree along the variables A, B, D, C for a cnf theory $\varphi = \{(\neg A \vee B), (\neg C \vee A), (A \vee B \vee D), C\}$. Hollow nodes and bars in the search tree represent illegal states, triangles represent solutions. The enclosed area corresponds to DPLL with unit-propagation.