# Directional consistency
# Chapter 4

ICS-275

Fall 2010

# Tractable classes

**Theorem 3.7.1** *1. The consistency binary constraint networks having no cycles can be decided by arc-consistent*

*2. The consistency of binary constraint networks with bi-valued domains can be decided by path-consistency,*

*3. The consistency of Horn cnf theories can be decided by unit propagation.*

# Backtrack-free search: or
## What level of consistency will guarantee global-consistency

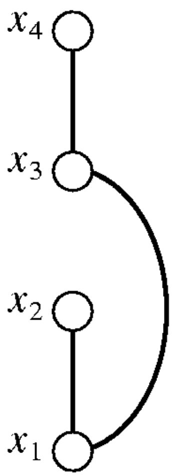
**Definition 4.1.1 (backtrack-free search)** *A constraint network is backtrack-free relative to a given ordering* $d = (x_1, ..., x_n)$ *if for every* $i \leq n$, *every partial solution of* $(x_1, ..., x_i)$ *can be consistently extended to include* $x_{i+1}$.

Backtrack free and queries:
Consistency,
All solutions
Counting
optimization

# Directional arc-consistency:

## another restriction on propagation

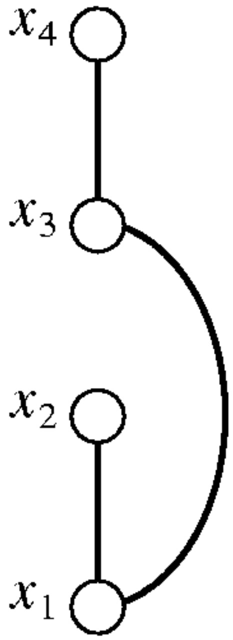
**Definition 4.3.1 (directional arc-consistency)** *A network is* directional-arc-consistent *relative to order* $d = (x_1, ..., x_n)$ *iff every variable* $x_i$ *is arc-consistent relative to every variable* $x_j$ *such that* $i \leq j$.




D4={white,blue,black}
D3={red,white,blue}
D2={green,white,black}
D1={red,white,black}
X1=x2, x1=x3,x3=x4

# Directional arc-consistency:
## another restriction on propagation

- D4={white,blue,black}
- D3={red,white,blue}
- D2={green,white,black}
- D1={red,white,black}
- X1=x2,
- x1=x3,
- x3=x4




After DAC:

- D1= {white},
- D2={green,white,black},
- D3={white,blue},
- D4={white,blue,black}

# Algorithm for directional arc-consistency (DAC)

DAC($\mathcal{R}$)

**Input**:A network $\mathcal{R} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$, its constraint graph $G$, and an ordering $d = (x_1, ...., x_n)$.

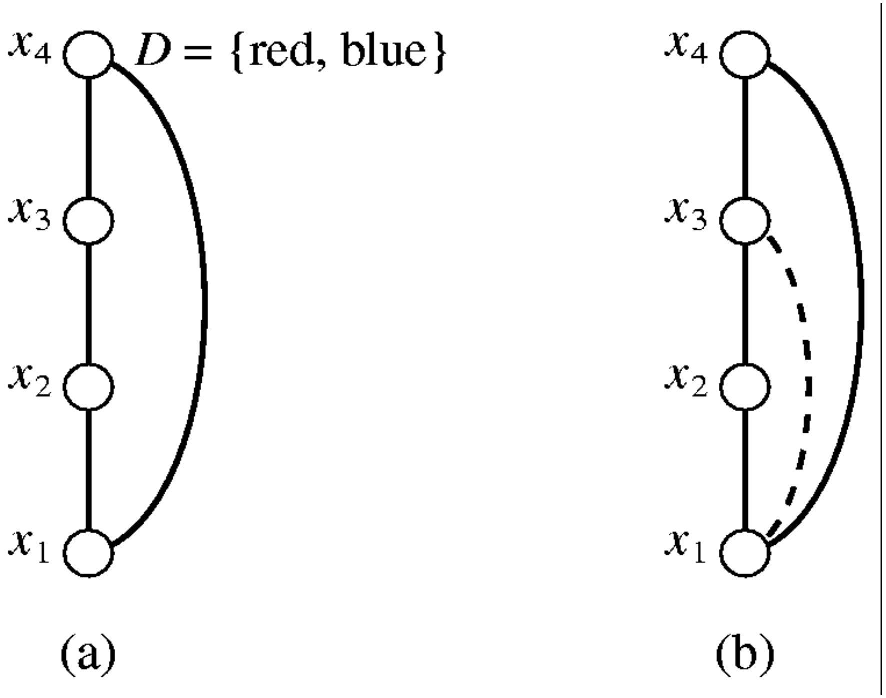
**Output:** A directional arc-consistent network.

1. **for** $i = n$ to 1 by $-1$ **do**
2. **for** each $j < i$ s.t. $R_{ji} \in \mathcal{R}$,
3. $D_j \leftarrow D_j \cap \pi_j(R_{ji} \bowtie D_i)$, (this is revise$((x_j), x_i)$).
4. **end-for**

Figure 4.6: Directional arc-consistency (DAC)

- Complexity: $O(ek^2)$

## Directional arc-consistency may not be enough → Directional path-consistency



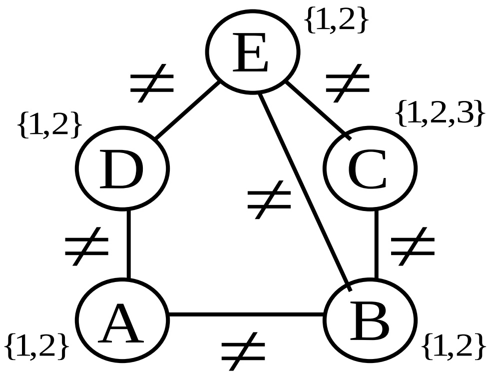

**Definition 4.3.5 (directional path-consistency)** *A network* $\mathcal{R}$ *is* directional path-consistent *relative to order* $d = (x_1, ..., x_n)$ *iff for every* $k \geq i, j$, *the pair* $\{x_i, x_j\}$ *is path-consistent relative to* $x_k$.

# Algorithm directional path consistency (DPC)

DPC($\mathcal{R}$)
**Input:** A binary network $\mathcal{R} = (X, D, C)$ and its constraint graph $G = (V, E)$, $d = (x_1, ...., x_n)$.
**Output:** A strong directional path-consistent network and its graph $G' = (V, E')$.
**Initialize:** $E' \leftarrow E$.
1. **for** $k = n$ to 1 by -1 **do**
2. (a) $\forall\, i \leq k$ such that $x_i$ is connected to $x_k$ in the graph, **do**
3. $D_i \leftarrow D_i \cap \pi_i(R_{ik} \bowtie D_k)$ ($Revise((x_i), x_k)$)
4. (b) $\forall\, i, j \leq k$ s.t. $(x_i, x_k)$,$(x_j, x_k) \in E'$ **do**
5. $R_{ij} \leftarrow R_{ij} \cap \pi_{ij}(R_{ik} \bowtie D_k \bowtie R_{kj})$ (Revise-3($(x_i, x_j), x_k$))
6. $E' \leftarrow E' \cup (x_i, x_j)$
7. **endfor**
8. **return** The revised constraint network $\mathcal{R}$ and $G' = (V, E')$.

**Theorem 4.3.7** *Given a binary network $\mathcal{R}$ and an ordering $d$, algorithm DPC generates a largest equivalent, strong, directional-path-consistent network relative to $d$. The time and space complexity of DPC is $O(n^3k^3)$, where $n$ is the number of variables and $k$ bounds the domain sizes.*

# Example of DPC



# Directional i-consistency

**Definition 4.3.8 (directional i-consistency)** *A network is* directional $i$-consistent *relative to order* $\mathbf{d} = (x_1, ..., x_n)$ *iff every* $i-1$ *variables are* $i$*-consistent relative to every variable that succeeds them in the ordering. A network is strong directional* $i$*-consistent if it is directional* $j$*-consistent for every* $j < i$.

# Algorithm directional i-consistency

**Directional i-consistency** $(DIC_i(\mathcal{R}))$

**Input:** a network $\mathcal{R} = (X, D, C)$, its constraint graph $G = (V, E)$, $d = (x_1, \ldots, x_n)$.

**output:** A strong directional $i$-consistent network along $d$ and its graph $G' = (V, E')$ .

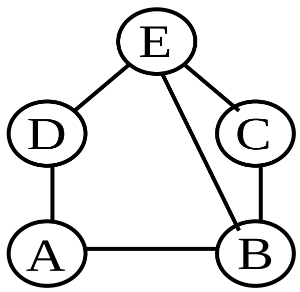
**Initialize:** $E' \leftarrow E$, $C' \leftarrow C$.

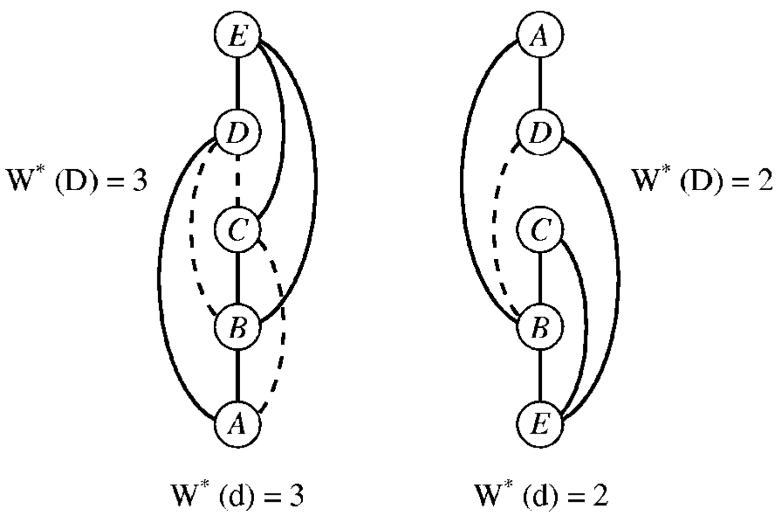
1. **for** $j = n$ to 1 by -1 **do**
2.   let $P = parents(x_j)$.
3.     if $|P| < i - 1$ then
4.       $Revise(P, x_j)$
5.     **else, for** each subset of $i - 1$ variables $S$, $S \subseteq P$, **do**
6.       Revise$(S, x_j)$
7.     **endfor**
8.     $C' \leftarrow C' \cup$ all generated constraints.
8.     $E' \leftarrow E' \cup \{(x_k, x_m) | x_k, x_m \in P\}$ (connect all parents of $x_j$)
9. **endfor.**
10. **return** $C'$ and $E'$.

Figure 4.9: Algorithm directional $i$-consistency $(DIC_i)$

# The induced-width

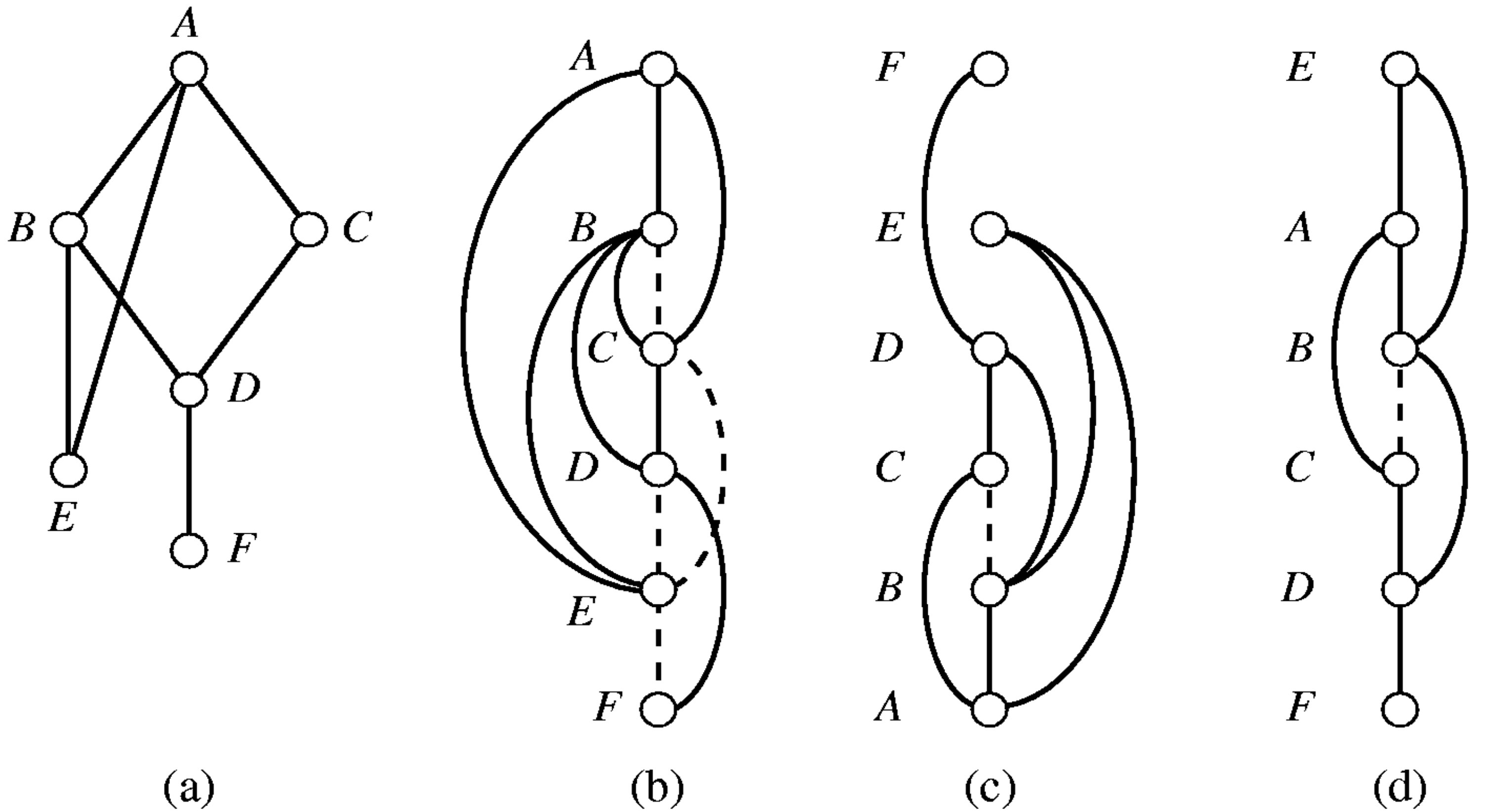
## DPC recursively connects parents in the ordered graph, yielding:




- **Width along ordering *d*, w(d):**
  - **max # of previous parents**
- Induced width w*(d):
  - **The width in the ordered *induced graph***
- Induced-width w*:
  - **Smallest induced-width over all orderings**
- Finding w*
  - **NP-complete *(Arnborg, 1985) but greedy heuristics (min-fill).***

# Induced-width



# Induced-width and DPC

- The induced graph of (G,d) is denoted (G*,d)
- The induced graph (G*,d) contains the graph generated by DPC along d, and the graph generated by directional i-consistency along d.

# Refined complexity using induced-width

**Theorem 4.3.11** *Given a binary network $\mathcal{R}$ and an ordering $\mathbf{d}$, the complexity of DPC along $\mathbf{d}$ is $O((w^*(d))^2 \cdot n \cdot k^3)$, where $w^*(\mathbf{d})$ is the induced width of the ordered constraint graph along $\mathbf{d}$.*

**Theorem 4.3.13** *Given a general constraint network $\mathcal{R}$ whose constraints' arity is bounded by $\mathbf{i}$, and an ordering $\mathbf{d}$, the complexity of $DIC_i$ along $\mathbf{d}$ is $O(n(w^*(d))^i \cdot (2k)^i)$.* □

- Consequently we wish to have ordering with minimal induced-width
- Induced-width is equal to tree-width to be defined later.
- Finding min induced-width ordering is NP-complete

# Greedy algorithms for induced-width

- Min-width ordering
- Max-cardinality ordering
- Min-fill ordering
- Chordal graphs

# Min-width ordering

MIN-WIDTH (MW)

**input:** a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
**output:** A min-width ordering of the nodes $d = (v_1, ..., v_n)$.
1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in $G$ with smallest degree.
3. put $r$ in position $j$ and $G \leftarrow G - r$.
   (Delete from $V$ node $r$ and from $E$ all its adjacent edges)
4. **endfor**

Figure 4.2: The min-width (MW) ordering procedure

# Min-induced-width

MIN-INDUCED-WIDTH (MIW)

**input:** a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
**output:** An ordering of the nodes $d = (v_1, ..., v_n)$.
1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in $V$ with smallest degree.
3. put $r$ in position $j$.
4. connect $r$'s neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5. remove $r$ from the resulting graph: $V \leftarrow V - \{r\}$.

Figure 4.3: The min-induced-width (MIW) procedure

# Min-fill algorithm

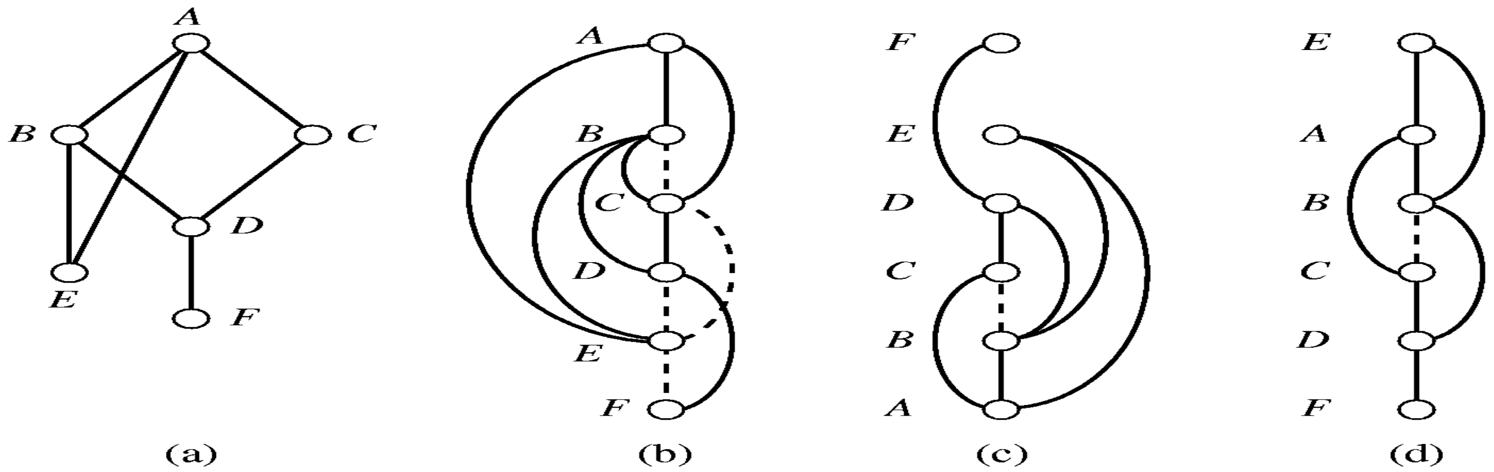
- Prefers a node who adds the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)

# Cordal graphs and max-cardinality ordering

- A graph is cordal if every cycle of length at least 4 has a chord
- Finding w* over chordal graph is easy using the max-cardinality ordering
- If G* is an induced graph it is chordal
- K-trees are special chordal graphs.
- Finding the max-clique in chordal graphs is easy (just enumerate all cliques in a max-cardinality ordering

# Example

We see again that *G* in Figure 4.1(a) is not chordal since the parents of *A* are not connected in the max-cardinality ordering in Figure 4.1(d). If we connect *B* and *C*, the resulting induced graph is chordal.



# Max-cardinality ordering

MAX-CARDINALITY (MC)

**input:** a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$

**output:** An ordering of the nodes $d = (v_1, ..., v_n)$.

1. Place an arbitrary node in position 0.
2. **for** $j = 1$ to $n$ do
3. $r \leftarrow$ a node in $G$ that is connected to a largest subset of nodes in positions 1 to $j - 1$, breaking ties arbitrarily.
4. **endfor**

**Figure 4.5 The max-cardinality (MC) ordering procedure.**

# Width vs local consistency: solving trees

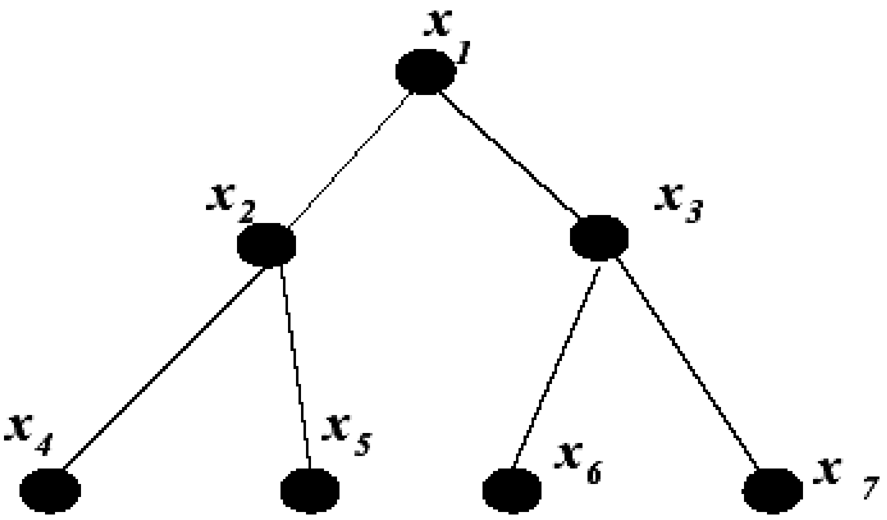



Figure 4.10: A tree network

**Theorem 4.4.1** *If a binary constraint network has a width of 1 and if it is arc-consistent, then it is backtrack-free along any width-1 ordering.*

# Tree-solving

**Tree-solving**

**Input**: A tree network $T = (X, D, C)$.

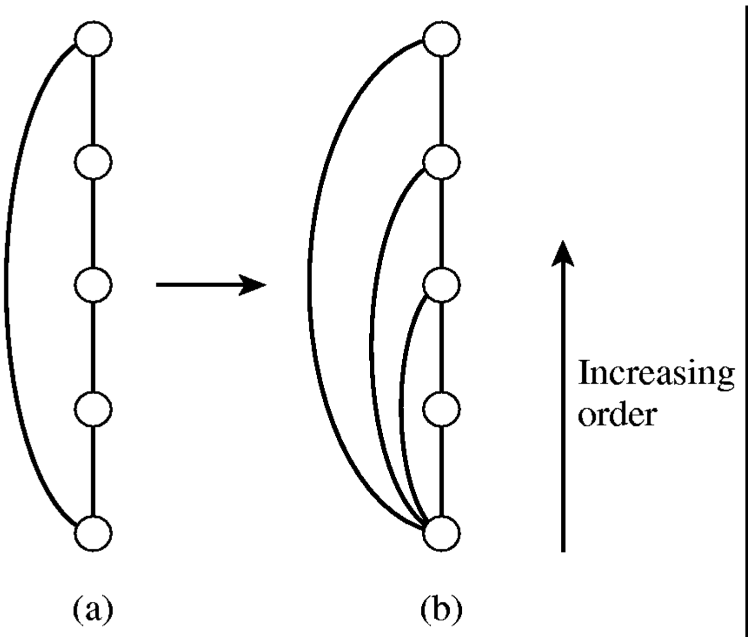
**Output:** A backtrack-free network along an ordering $d$.

1. generate a width-1 ordering, $d = x_1, \ldots, x_n$.
2. **let** $x_{p(i)}$ denote the parent of $x_i$ in the rooted ordered tree.
3. **for** $i = n$ to 1 **do**
4.     *Revise* $((x_{p(i)}), x_i)$;
5.     **if** the domain of $x_{p(i)}$ is empty, exit. (no solution exists).
6. **endfor**

Figure 4.11: Tree-solving algorithm

$$complexity : O(nk^2)$$

# Width-2 and DPC




**Theorem 4.4.3 (Width-2 and directional path-consistency)** *If $\mathcal{R}$ is directional arc and path-consistent along* $\mathbf{d}$*, and if it also has width-2 along* $\mathbf{d}$*, then it is backtrack-free along* $\mathbf{d}$*.* □

# Width vs directional consistency
## (Freuder 82)

**Theorem 4.4.5 (Width (i-1) and directional i-consistency)** *Given a general network $\mathcal{R}$, its ordered constraint graph along* $\mathbf{d}$ *has a width of* $\mathbf{i}-1$ *and if it is also strong directional* $\mathbf{i}$*-consistent, then* $\mathcal{R}$ *is backtrack-free along* $\mathbf{d}$.

# Width vs i-consistency

- DAC and width-1
- DPC and width-2
- DIC_i and with-(i-1)
- → backtrack-free representation

- If a problem has width 2, will DPC make it backtrack-free?
- **Adaptive-consistency**: applies i-consistency when i is adapted to the number of parents

# Adaptive-consistency

ADAPTIVE-CONSISTENCY (AC1)

**Input:** a constraint network $\mathcal{R} = (X, D, C)$, its constraint graph $G = (V, E)$, $d = (x_1, \ldots, x_n)$.

**output:** A backtrack-free network along $d$

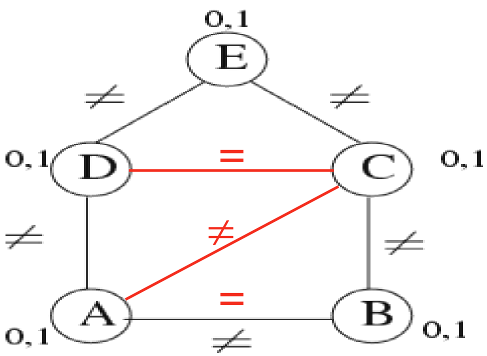
**Initialize:** $C' \leftarrow C$, $E' \leftarrow E$

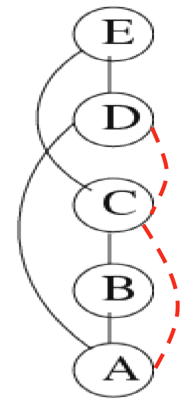
1. **for** $j = n$ to 1 do
2. Let $S \leftarrow parents(x_j)$.
3. $R_S \leftarrow Revise(S, x_j)$ (generate all partial solutions over $S$ that can extend to $x_j$).
4. $C' \leftarrow C' \cup R_S$
5. $E' \leftarrow E' \cup \{(x_k, x_r) | x_k, x_r \in parents(x_j)\}$ (connect all parents of $x_j$)
5. **endfor**.

Figure 4.13: Algorithm adaptive-consistency– version 1

# Bucket Elimination

## Adaptive Consistency (Dechter & Pearl, 1987)




Bucket E: $E \neq D,\ E \neq C$

Bucket D: $D \neq A$ → $D = C$

Bucket C: $C \neq B$ → $A \neq C$

Bucket B: $B \neq A$ → $B = A$

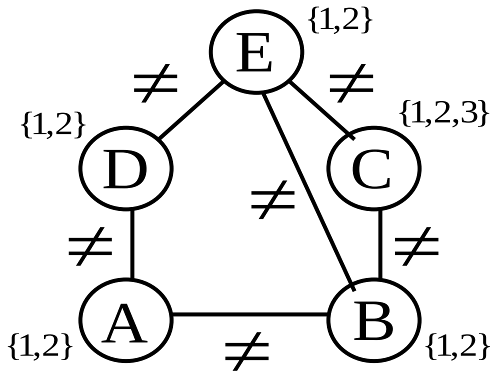
Bucket A: → contradiction

**Complexity:** $O(n\ exp(w^{*}))$

$w^{*}$ - ***induced width***

# Bucket Elimination

## Adaptive Consistency (Dechter & Pearl, 1987)



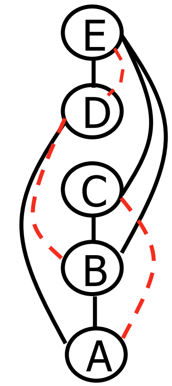

$Bucket(E):\ E \neq D,\ E \neq C,\ E \neq B$

$Bucket(D):\ D \neq A\ ||\ R_{DCB}$

$Bucket(C):\ C \neq B\ ||\ R_{ACB}$

$Bucket(B):\ B \neq A\ ||\ R_{AB}$

$Bucket(A):\ R_A$



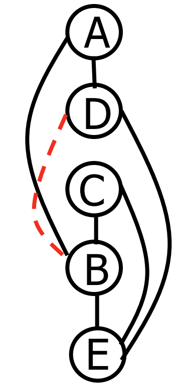

$Bucket(A):\ A \neq D,\ A \neq B$

$Bucket(D):\ D \neq E\ ||\ R_{DB}$

$Bucket(C):\ C \neq B,\ C \neq E$

$Bucket(B):\ B \neq E\ ||\ R^{D}{}_{BE},\ R^{C}{}_{BE}$

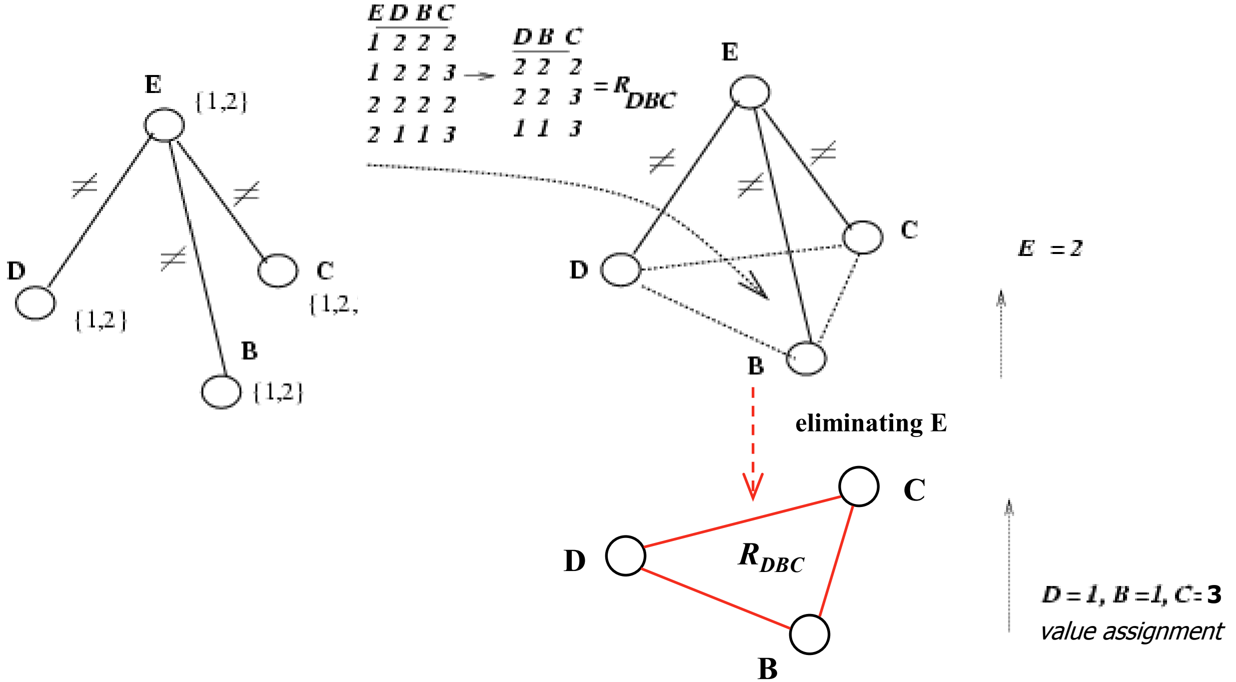
$Bucket(E):\ ||\ R_E$




**Time and space:** $O(n\ exp(w^*(d)))$,

$w^*(d)$ - ***induced width along ordering d***

# The Idea of Elimination



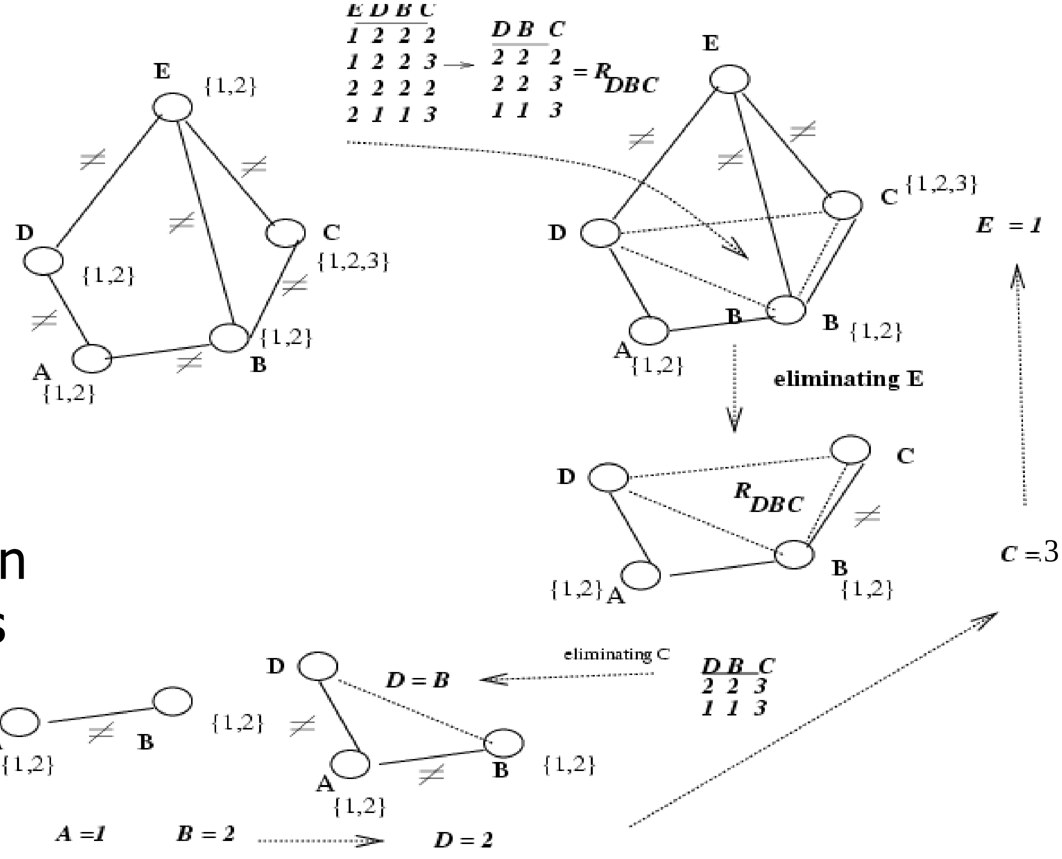

$$R_{DBC} = \prod_{DBC} R_{ED} \bowtie R_{EB} \bowtie R_{EC}$$

Eliminate variable E $\Leftrightarrow$ join and project

# Variable Elimination

Eliminate variables one by one: "constraint propagation"

Solution generation after elimination is backtrack-free



# Adaptive-consistency, bucket-elimination

ADAPTIVE-CONSISTENCY (AC)

**Input:** a constraint network $\mathcal{R}$, an ordering $d = (x_1, \ldots, x_n)$
**output:** A backtrack-free network, denoted $E_d(\mathcal{R})$, along $d$, if the empty constraint was not generated. Else, the problem is inconsistent

1. Partition constraints into $bucket_1, \ldots, bucket_n$ as follows:
   **for** $i \leftarrow n$ **downto** 1, put in $bucket_i$ all unplaced constraints mentioning $x_i$.
2. **for** $p \leftarrow n$ **downto** 1 **do**
3. **for** all the constraints $R_{S_1}, \ldots, R_{S_j}$ in $bucket_p$ **do**
4. $A \leftarrow \bigcup_{i=1}^{j} S_i - \{x_p\}$
5. $R_A \leftarrow \Pi_A(\bowtie_{i=1}^{j} R_{S_i})$
6. **if** $R_A$ is not the empty relation **then** add $R_A$ to the bucket of the latest variable in scope $A$,
7. **else** exit and return the empty network
8. **return** $E_d(\mathcal{R}) = (X, D, bucket_1 \cup bucket_2 \cup \cdots \cup bucket_n)$

Figure 4.14: Adaptive-Consistency as a bucket-elimination algorithm

# Properties of bucket-elimination (adaptive consistency)

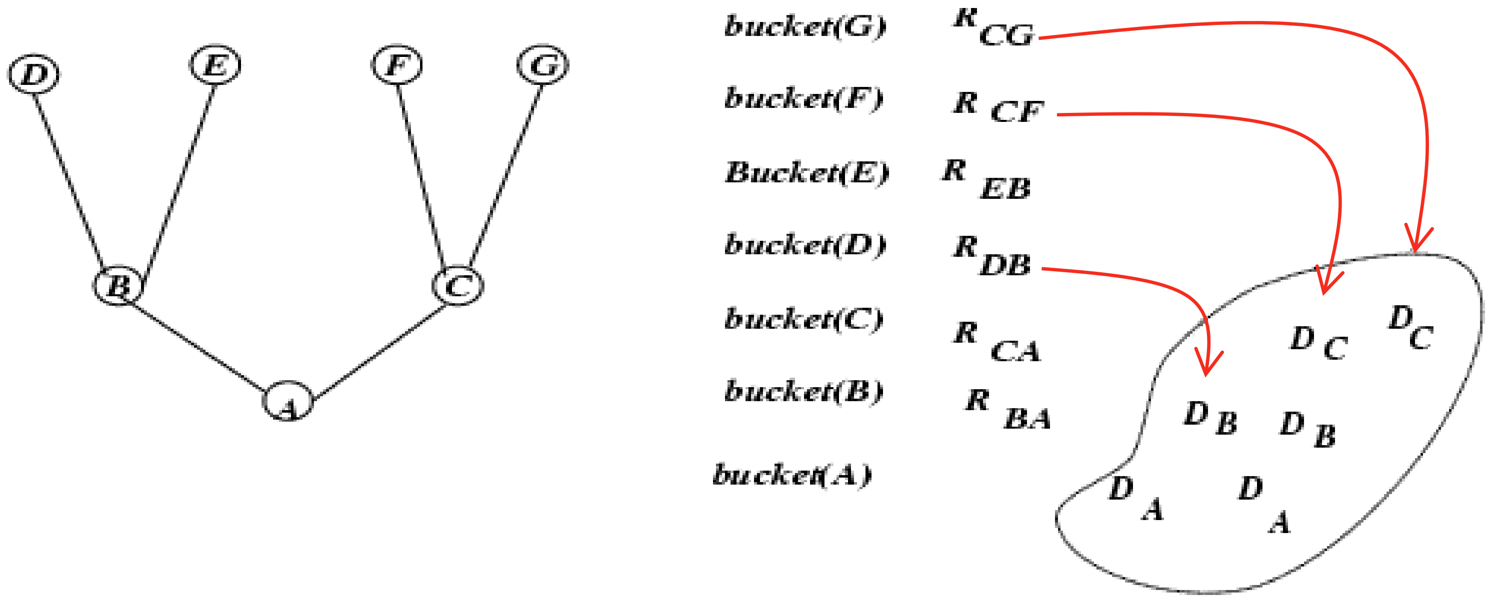
- **Adaptive consistency generates a constraint network that is backtrack-free (can be solved without dead-ends).**
- **The time and space complexity of adaptive consistency along ordering *d* is** $O(n\,(2k)^{w^*+1}), O(n\,(k)^{w^*+1})$ **respectively, or O(r k^(w*+1)) when r is the number of constraints.**
- **Therefore, problems having bounded induced width are tractable (solved in polynomial time)**
- **Special cases: *trees* ( w*=1 ), *series-parallel networks* (w*=2 ), and in general *k-trees* ( w*=k ).**

# Back to Induced width

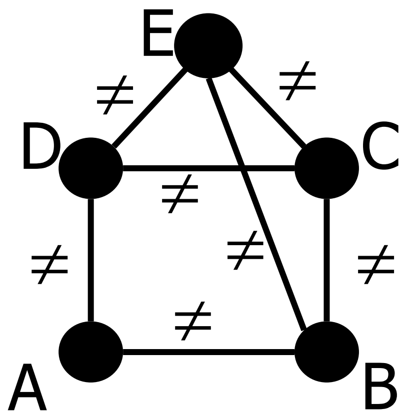
- Finding minimum-w* ordering is NP-complete (Arnborg, 1985)
- Greedy ordering heuristics: *min-width, min-degree, max-cardinality* (Bertele and Briochi, 1972; Freuder 1982), Min-fill.

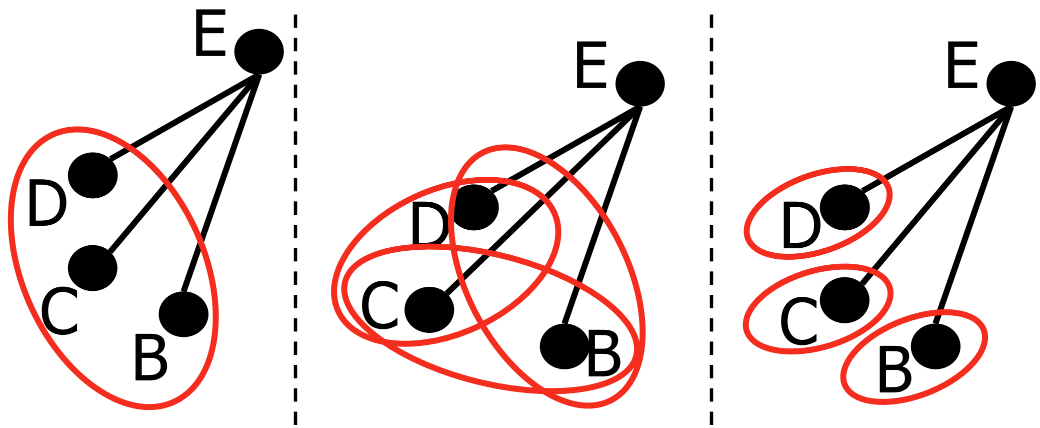
# Solving Trees
**(Mackworth and Freuder, 1985)**

Adaptive consistency is linear for trees and equivalent to enforcing directional arc-consistency (recording only unary constraints)



# Summary: directional i-consistency




| Adaptive | d-path | d-arc |
|---|---|---|
| $R_{DCB}$ | $R_{DC}, R_{DB}$<br>$R_{CB}$ | $R_D$<br>$R_C$<br>$R_D$ |

**E : E $\neq$ D, E $\neq$ C, E $\neq$ B**

**D : D $\neq$ C, D $\neq$ A**

**C : C $\neq$ B**

**B : A $\neq$ B**

**A :**

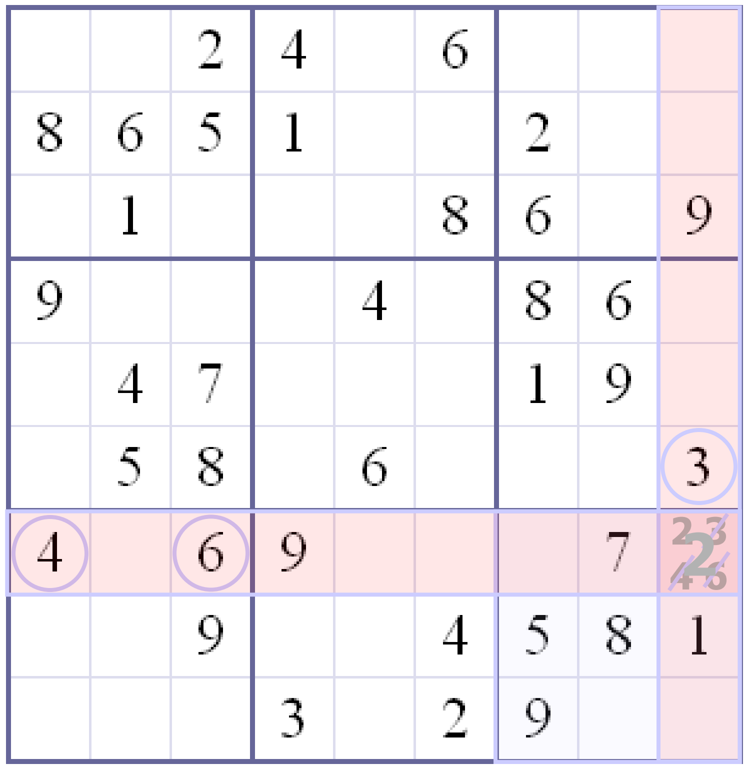
# Relational consistency (Chapter 8)

- Relational arc-consistency
- Relational path-consistency
- Relational m-consistency
- Relational consistency for Boolean and linear constraints:
  - Unit-resolution is relational-arc-consistency
  - Pair-wise resolution is relational path-consistency

# Sudoku's propagation

- http://www.websudoku.com/
- What kind of propagation we do?

# Sudoku



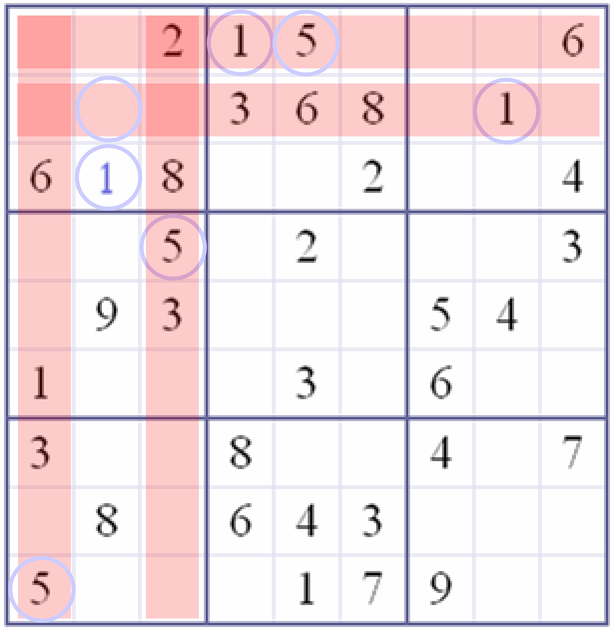

- Variables: 81 slots
- Domains = {1,2,3,4,5,6,7,8,9}
- Constraints:
  - 27 not-equal

**Constraint propagation**

**Each row, column and major block must be alldifferent**

**“Well posed” if it has unique solution: 27 constraints**

# Sudoku



**Each row, column and major block must be alldifferent**

**“Well posed” if it has unique solution**