# Tree Decomposition methods

## Chapter 9

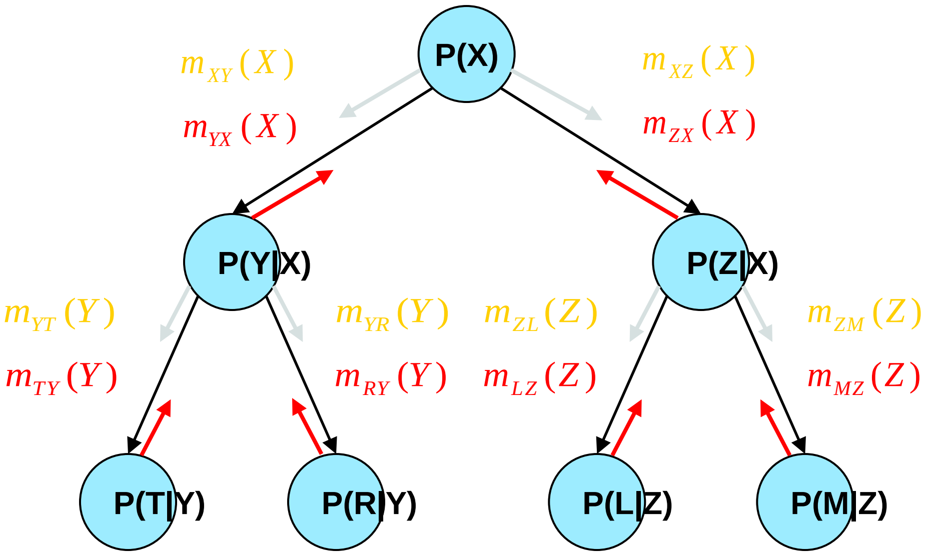
ICS-275

Spring 2010

## Tree-solving

**Belief updating (sum-prod)**

**CSP – consistency (projection-join)**

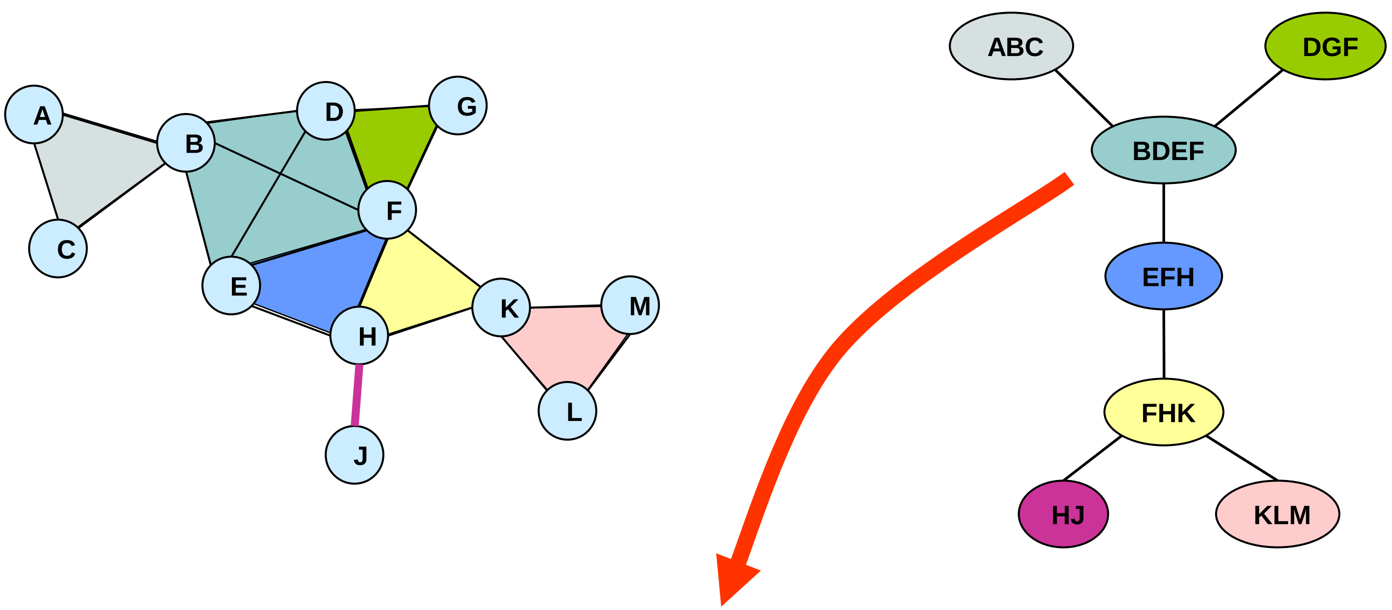



**MPE (max-prod)**

**#CSP (sum-prod)**

**Trees are processed in linear time and memory**

## Inference and Treewidth




*treewidth* = 4 - 1 = 3

treewidth = (maximum cluster size) - 1

# The Dual problem/The acyclic problem

The dual graph of a constraint problem associates a node with each constraint scope and an arc for each two nodes sharing variables. This transforms a non-binary constraint problem into a binary one, called the *dual problem:*

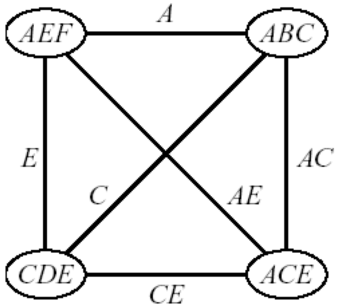
Variables: constraints,
Domains: legal tuples of the relation
Binary constraints between any two dual variables that share original variables, enforcing equality on the values assigned to the shared variables.

Therefore, if a problem's dual graph happens to be a tree, it can be solved by the tree-solving algorithm.

It turns out, however, that sometimes, even when the dual graph does not look like a

# Dual Constraint Problems

- Constraints can be: C= AVE
- F=AVE and so on…




(c)

(d)

# Dual Constraint Problems

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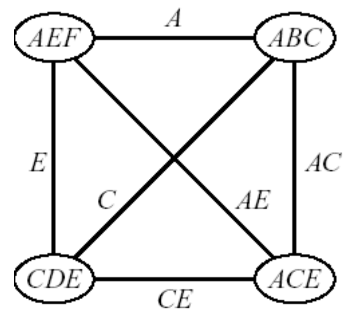
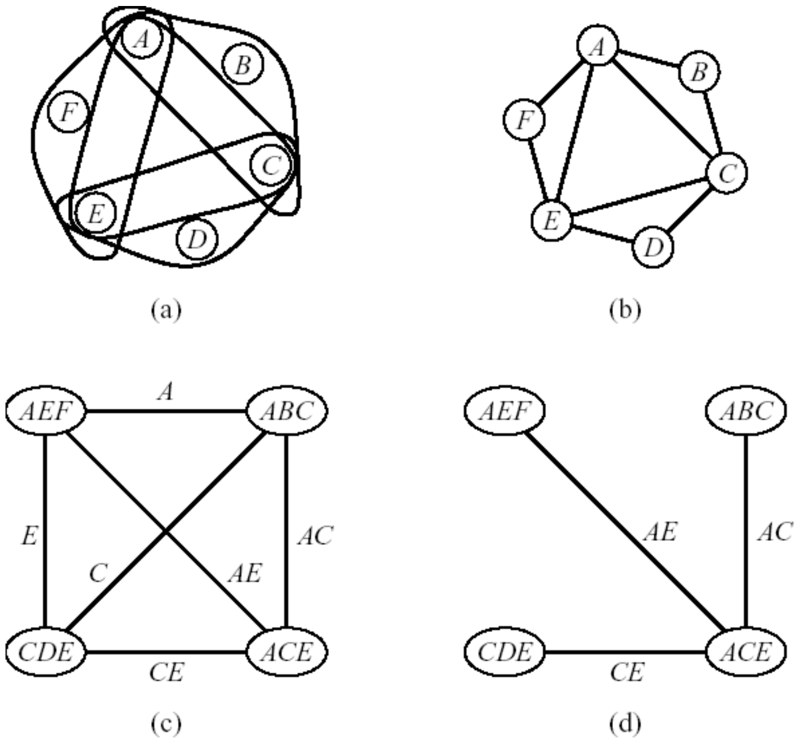
- F=AVE and so on…




(c)

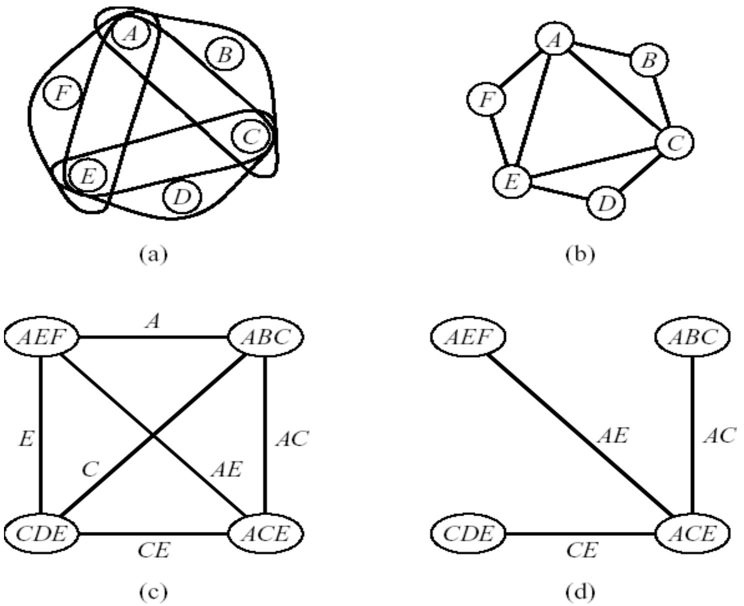
# Graph concepts reviews: Hyper graphs and dual graphs

- **A hypergraph** is H = (V,S) , V= {v_1,..,v_n} and a set of subsets **Hyperegdes**: S={S_1, ..., S_l }.
- **Dual graphs** of a hypergaph: The nodes are the hyperedges and a pair of nodes is connected if they share vertices in V. The arc is labeled by the shared vertices.
- **A primal graph** of a hypergraph H = (V,S) has V as its nodes, and any two nodes are connected by an arc if they appear in the same hyperedge.
- if all the constraints of a network R are binary, then its hypergraph is identical to its primal graph.



# Acyclic Networks

- **The running intersection property (connectedness):** An arc can be removed from the dual graph if the variables labeling the arcs are shared along an alternative path between the two endpoints.
- **Join graph**: An arc subgraph of the dual graph that satisfies the connectedness property.
- **Join-tree:** a join-graph with no cycles
- **Hypertree:** A hypergraph whose dual graph has a join-tree.
- **Acyclic network:** is one whose hypergraph is a hypertree.



# Example

- Constraints are:
- *R_{ABC} = R_{AEF} = R_{CDE} = {(0,0,1) (0,1,0)(1,0,0)}*
- *R_{ACE} = { (1,1,0) (0,1,1) (1,0,1) }.*

- d= (R_{ACE},R_{CDE},R_{AEF},R_{ABC}).
  - When processing R_{ABC}, its parent relation is R_{ACE};

$$R_{ACE} = \pi_{ACE}(R_{ACE} \otimes R_{ABC}) = \{(0,1,1)(1,0,1)\}$$

  - processing R_{AEF} we generate relation

$$R_{ACE} = \pi_{ACE}(R_{ACE} \otimes R_{AEF}) = \{(0,1,1)\}$$

    - processing R_{CDE} we generate:
    - R_{ACE} = \pi_{ACE} ( R_{ACE} x R_{CDE} ) = {(0,1,1)}.
- A solution is generated by picking the only allowed tuple for R_{ACE}, A=0,C=1,E=1, extending it with a value for D that satisfies R_{CDE}, which is only D=0, and then similarly extending the assignment to F=0 and B=0, to satisfy R_{AEF} and R_{ABC}.

# Solving acyclic networks

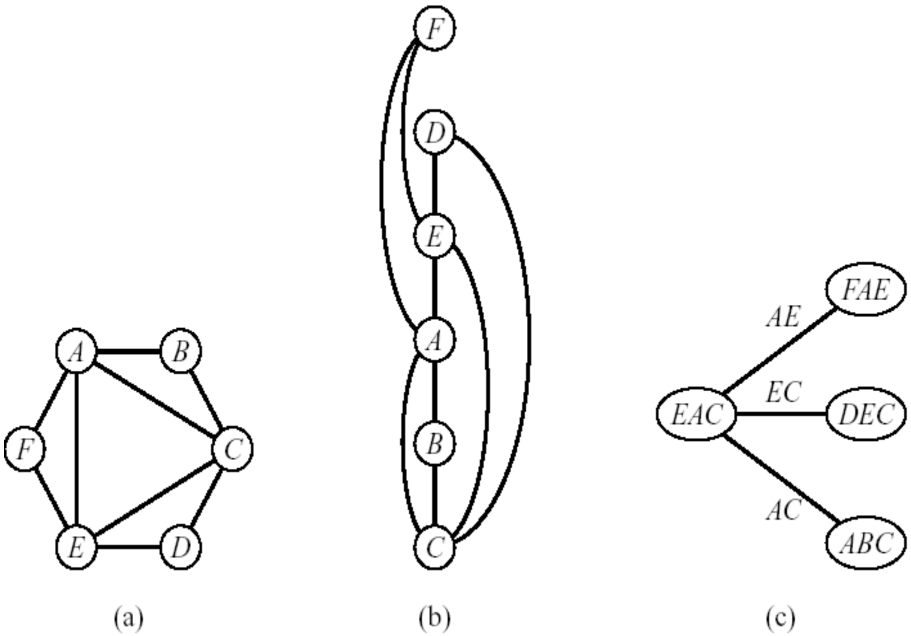
- Algorithm ***<u>acyclic-solving</u>*** applies a tree algorithm to the join-tree. It applies (a little more than) directional relational arc-consistency from leaves to root.

- **Complexity**: acyclic-solving is O(r l  log l) steps, where r is the number of constraints and l bounds the number of tuples in each constraint relation
- (It assumes that join of two relations when one's scope is a subset of the other can be done in linear time)

# Recognizing acyclic networks

- **Dual-based recognition**:
  - perform maximal spanning tree over the dual graph and check connectedness of the resulting tree.
  - Dual-acyclicity complexity is O(e^3)
- **Primal-based recognition**:
  - **Theorem** (Maier 83): A hypergraph has a join-tree iff its primal graph is chordal and conformal.
  - A chordal primal graph is **conformal** relative to a constraint hypergraph iff there is a one-to-one mapping between maximal cliques and scopes of constraints.

# Primal-based recognition




- Check cordality using max-cardinality ordering.
- Test conformality
- Create a join-tree: connect every clique to an earlier clique sharing maximal number of variables

# Tree-based clustering

- Convert a constraint problem to an acyclic-one: group subset of constraints to clusters until we get an acyclic problem.

- **Tree-decomposition**: Hypertree embedding of a hypergraph *H* = (X,H) is a hypertree *S* = (X, S) s.t., for every h in H there is h_1 in S s.t. h is included in h_1.

- This yield algorithm join-tree clustering and tree-decomposition in general

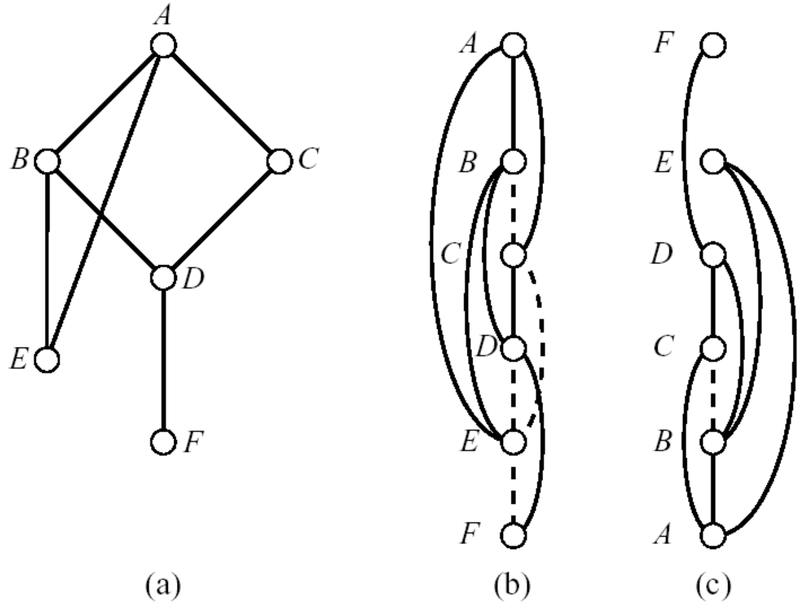
- **Hypertree decomposition**: Hypertree partitioning of a hypergraph *H* = (X,H) is a hypertree *S* = (X, S) s.t., for every h in H there is h_1 in S s.t. h is included in h_1 **and X is the union of scopes in h_1.**

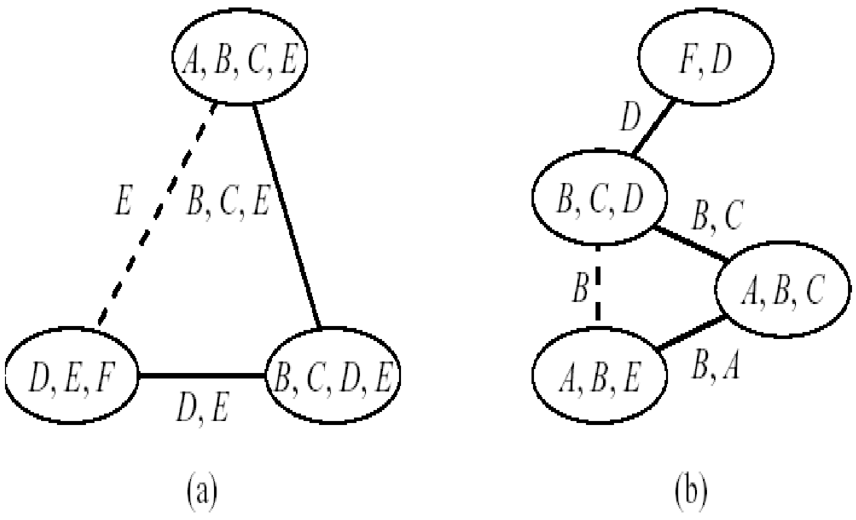
# Join-tree clustering

- **Input**: A constraint problem R =(X,D,C) and its primal graph G = (X,E).
- **Output**: An equivalent acyclic constraint problem and its join-tree*: T*= (X,D, {C '})
- 1. Select an d = (x_1,...,x_n)
- 2. **Triangulation**(create the induced graph along $d$ and call it G^*: )
- for j=n to 1 by -1 do
- E ← E U {(i,k)| (i,j) in E,(k,j) in E }
- 3. **Create a join-tree of the induced graph G^*:**
- a. Identify all maximal cliques (each variable and its parents is a clique).
- Let C_1,...,C_t be all such cliques,
- b. Create a tree-structure T over the cliques:
- Connect each C_{i} to a C_{j} (j < I) with whom it shares largest subset of variables.
- 4. Place each input constraint in one clique containing its scope, and let
- P_i be the constraint subproblem associated with C_i.
- 5. Solve P_i and let {R'}_i $ be its set of solutions.
- 6. Return C' = {R'}_1,..., {R'}_t the new set of constraints and their join-tree, T.

- **Theorem: join-tree clustering transforms a constraint network into an acyclic network**

# Example of tree-clustering



# Complexity of JTC

- **complexity of** JTC**:** *join-tree clustering is* $O(r\,k^{\wedge}(w^*(d)+1))$ time and $O(nk^{\wedge}(w^*(d)+1))$ space, *where k is the maximum domain size and* $w^*(d)$ *is the induced width of the ordered graph.*
- *The complexity of acyclic-solving is* $O(n\,w^*(d)\,(\log k)\,k^{\wedge}(w^*(d)+1))$

# Unifying tree-decompositions

Let *R=<X,D,C>* be a CSP problem. A tree decomposition for R is $<T,\chi,\psi>$, such that

- T=(V,E) is a tree
- $\chi$ associates a set of variables $\chi(v) \subseteq X$ with each node v
- $\psi$ associates a set of functions $\psi(v) \subseteq C$ with each node v

such that

- 1. $\forall R_i \in C$, there is exactly one v such that $R_i \in \psi(v)$ and $scope(R_i) \subseteq \chi(v)$.
- 2. $\forall x \in X$, the set $\{v \in V | x \subseteq \chi(v)\}$ induces a connected subtree.

# HyperTree Decomposition

Let *R=<X,D,C>* be CSP problem. A tree decomposition is $<T,\chi,\psi>$, such that

- T=(V,E) is a tree
- $\chi$ associates a set of variables $\chi(v) \subseteq X$ with each node
- $\psi$ associates a set of functions $\psi(v) \subseteq C$ with each node

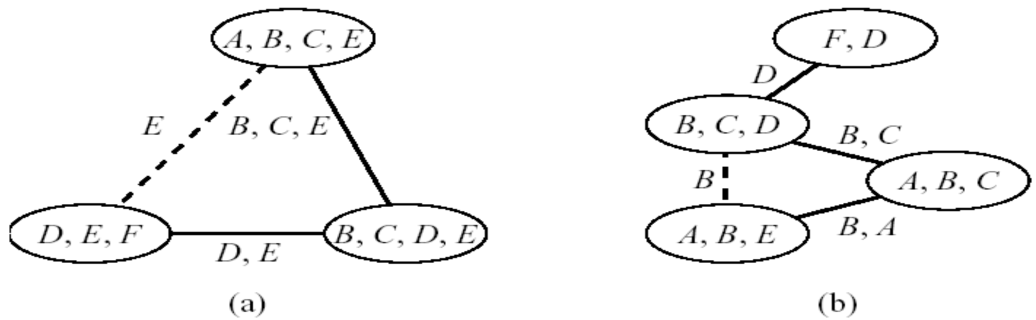
such that

- 1. $\forall R_i \in C$, there is exactly one v such that $R_i \in \psi(v)$ and $scope(R_i) \subseteq \chi(v)$.
- 1a. $\forall v$, $\chi(v) \subseteq scope(\psi(v))$.
- 2. $\forall x \in X$, the set $\{v \in V | x \subseteq \chi(v)\}$ induces a connected subtree.

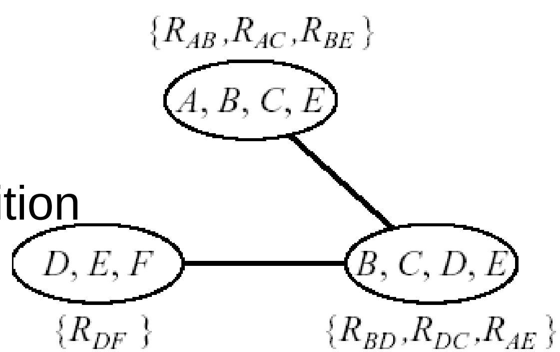
**w (tree-width) =** $\max_{v \in V} |\chi(v)|$
**hw (hypertree width) =** $\max_{v \in V} |\psi(v)|$

**sep (max separator size) =** $\max_{(u,v)} |sep(u,v)|$

# Example of two join-trees again




Tree decomposition

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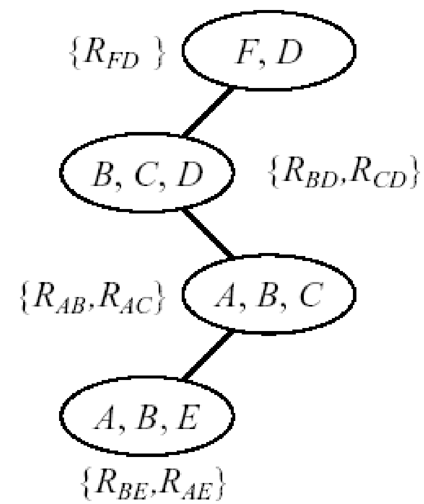



(b)

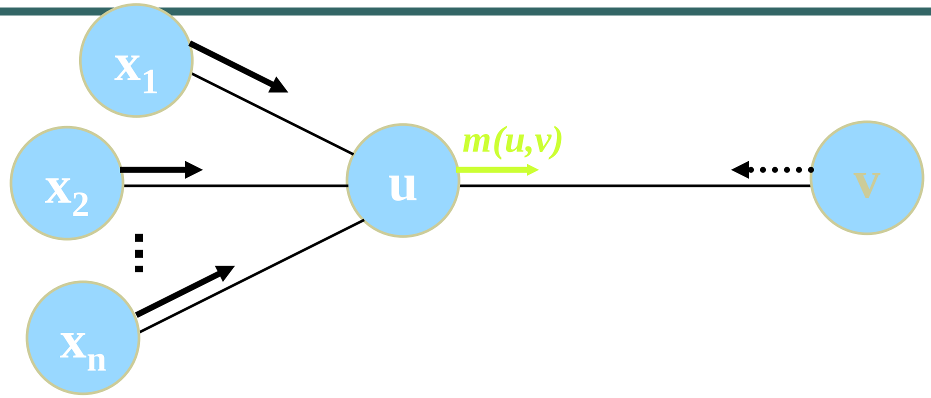
hyperTree-decomposition

# Cluster Tree Elimination

- Cluster Tree Elimination (CTE) works by passing messages along a tree-decomposition

- Basic idea:
  - Each node sends one message to each of its neighbors
  - Node *u* sends a message to its neighbor *v* only when *u* received messages from all its other neighbors

# Constraint Propagation

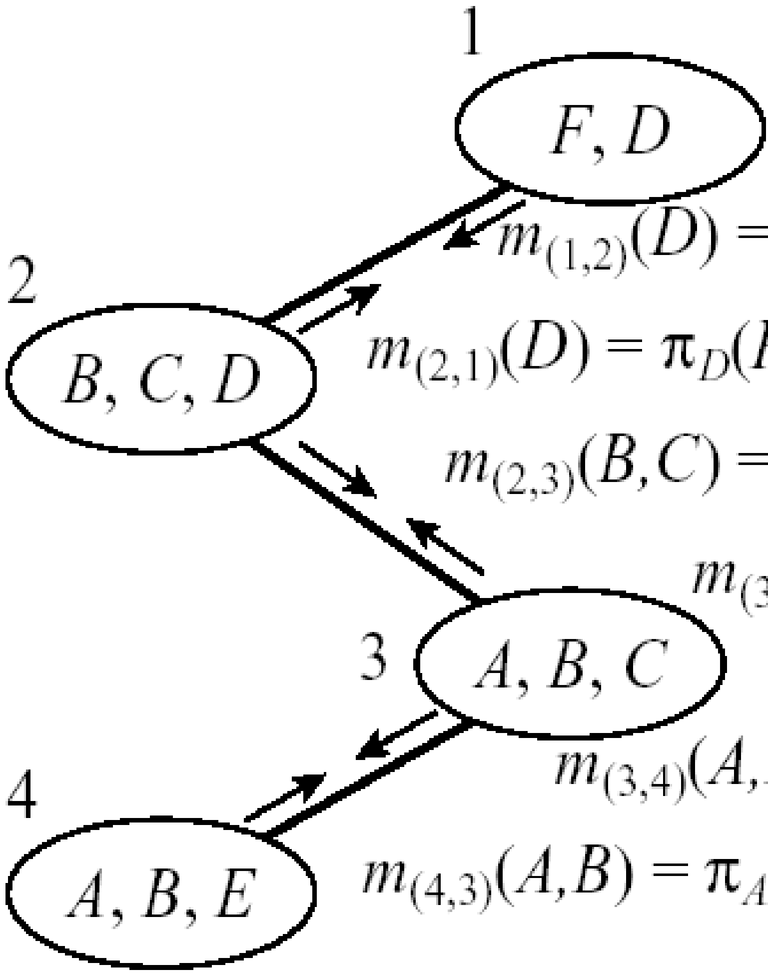



$$cluster(u) = \psi(u) \cup \{m(x_1,u), m(x_2,u), \ldots, m(x_n,u), m(v,u)\}$$

Compute the message :

$$m_{(u,v)} = \pi_{sep(u,v)}(\otimes_{R_i \in cluster(u)} R_i)$$

# Example of CTE message propagation




$$m_{(1,2)}(D) = \pi_D(R_{FD})$$

$$m_{(2,1)}(D) = \pi_D(R_{BD} \bowtie R_{CD} \bowtie m_{(3,2)})$$

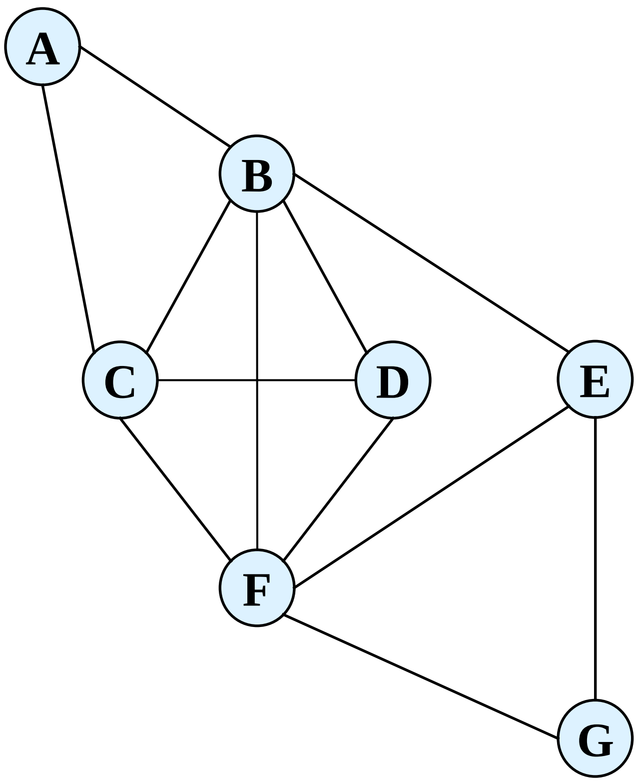
$$m_{(2,3)}(B,C) = \pi_{BC}(R_{BD} \bowtie R_{CD} \bowtie m_{(1,2)})$$

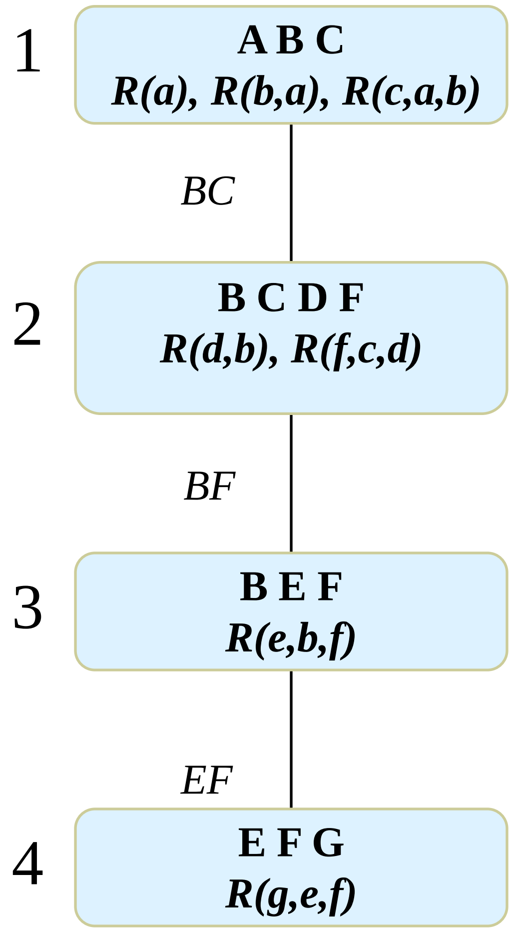
$$m_{(3,2)}(B,C) = \pi_{BC}(R_{AB} \bowtie R_{AC} \bowtie m_{(4,3)})$$

$$m_{(3,4)}(A,B) = \pi_{AB}(R_{AB} \bowtie R_{AC} \bowtie m_{(2,3)})$$

$$m_{(4,3)}(A,B) = \pi_{AB}(R_{BE} \bowtie R_{AE})$$

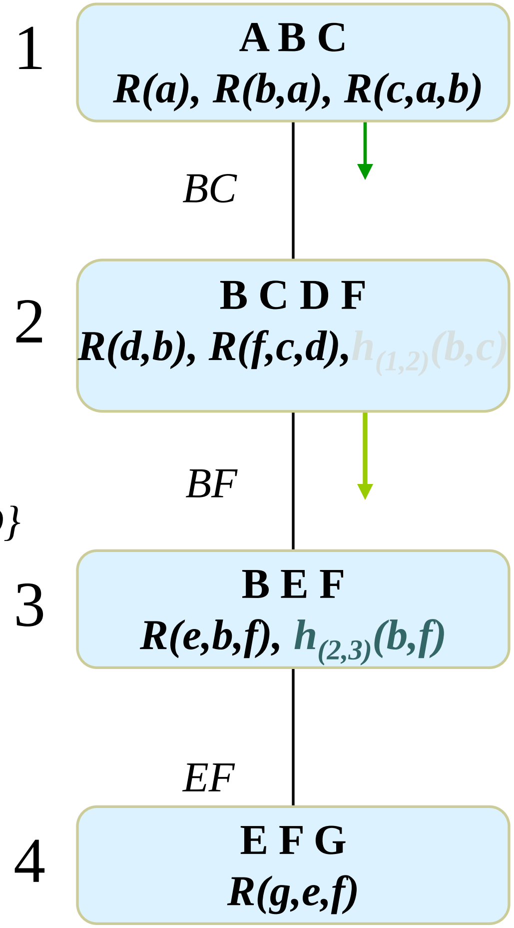
## Join-Tree Decomposition
**(Dechter and Pearl 1989)**




- Each function in a cluster
- Satisfy running intersection property
- Tree-width: number of variables in a cluster-1
- Equals induced-width

## Cluster Tree Elimination




*sep(2,3)={B,F}*

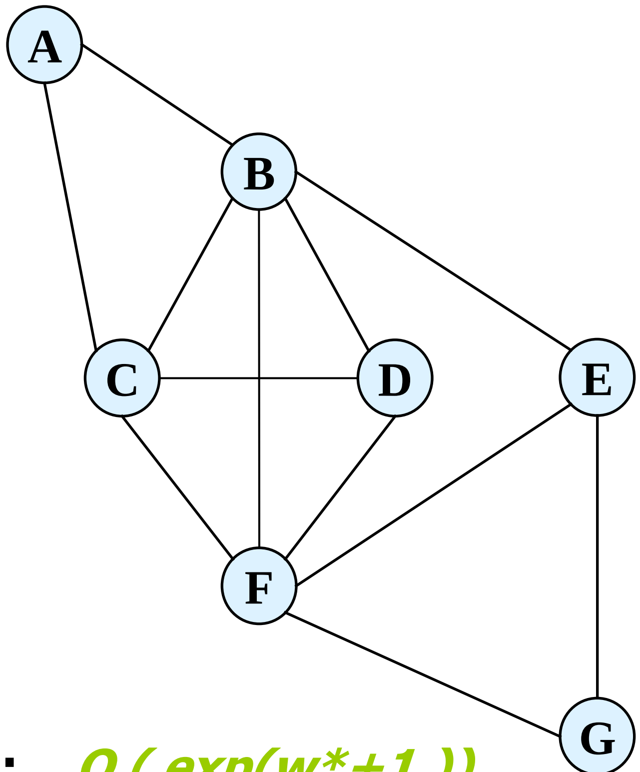
*elim(2,3)={C,D}*

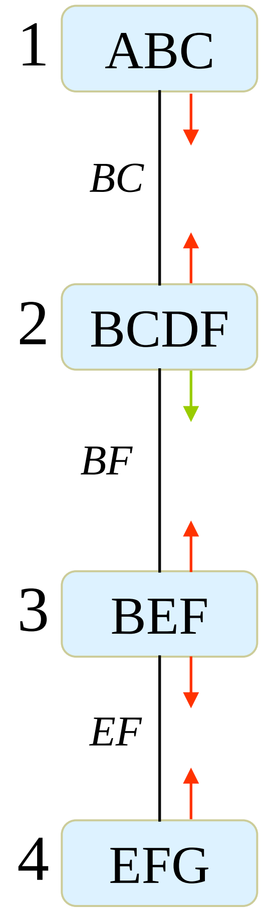
project join

$$h_{(1,2)}(b,c) = \Downarrow_{a} R(a) \otimes R(b,a) \otimes R(c,a,b)$$

$$h_{(2,3)}(b,f) = \Downarrow_{c,d} R(d,b) \otimes R(f,c,d) \otimes h_{(1,2)}(b,c)$$

## CTE: Cluster Tree Elimination




$$h_{(1,2)}(b,c) = \Downarrow_a R(a) \otimes R(b,a) \otimes R(c,a,b)$$

$$h_{(2,1)}(b,c) = \Downarrow_{d,f} R(d,b) \otimes R(f,c,d) \otimes h_{(3,2)}(b,f)$$

$$h_{(2,3)}(b,f) = \Downarrow_{c,d} R(d,b) \otimes R(f,c,d) \otimes h_{(1,2)}(b,c)$$

$$h_{(3,2)}(b,f) = \Downarrow_e R(e,b,f) \otimes h_{(4,3)}(e,f)$$

$$h_{(3,4)}(e,f) = \Downarrow_b R(e,b,f) \otimes h_{(2,3)}(b,f)$$

$$h_{(4,3)}(e,f) = \Downarrow_g R(g,e,f)$$

**Time:** ***O ( exp(w*+1 ))***
**Space:** ***O ( exp(sep))***
***Time: O(exp(hw)) (Gottlob et. Al., 2000)***

# Cluster Tree Elimination - properties

- Correctness and completeness: Algorithm CTE is sound and complete for generating minimal subproblems over chi(v) for every v: i.e. the solution of each subproblem is minimal.
- Time complexity: $O(\,deg \times (r+N) \times k^{w^*+1}\,)$
- Space complexity: $O(\,N \times d^{\,sep})$

where *deg* = the maximum degree of a node

r = number of of CPTs

*N* = number of nodes in the tree decomposition

*k*= the maximum domain size of a variable

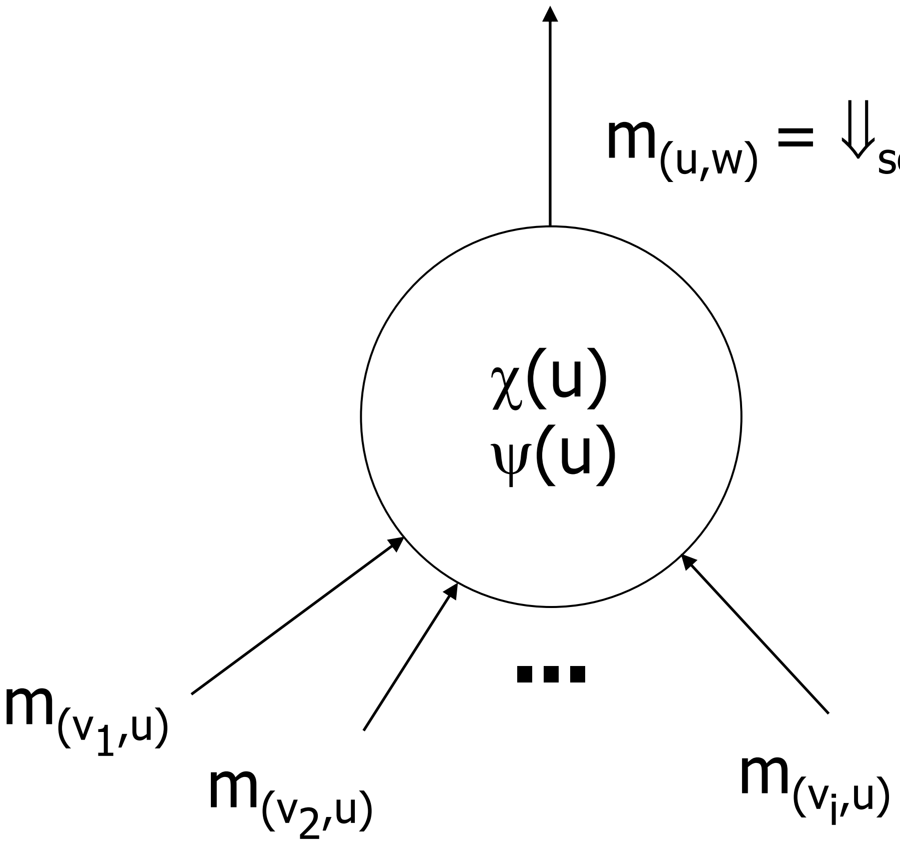
*w** = the induced width

*sep* = the separator size

- JTC is $O(\,r \times k^{\,w^*+1}\,)$ *time and space*

# Cluster-Tree Elimination (CTE)

$$m_{(u,w)} = \Downarrow_{sep(u,w)} [\otimes_j \{m_{(v_j,u)}\} \otimes \psi(u)]$$



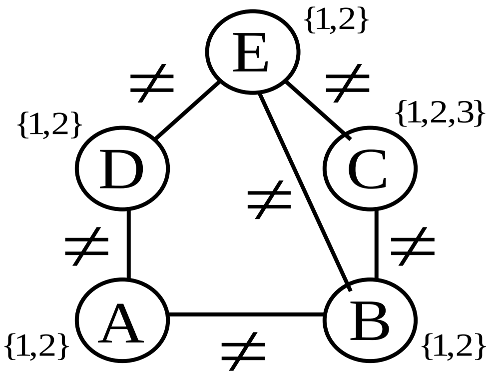
## Adaptive-consistency as tree-decomposition

- Adaptive consistency is a message-passing along a bucket-tree

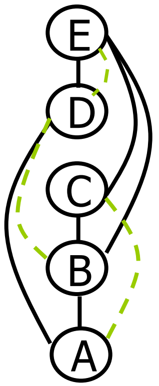
- **Bucket trees**: each bucket is a node and it is connected to a bucket to which its message is sent.
  - The variables are the clicue of the triangulated graph
  - The functions are those placed in the initial partition

# Bucket Elimination

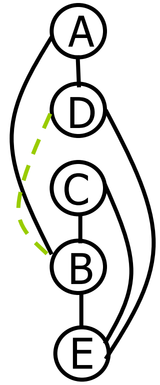
**Adaptive Consistency** **(Dechter and Pearl, 1987)**




$$Bucket(E):\ \mathrm{E} \neq \mathrm{D},\ \mathrm{E} \neq \mathrm{C},\ \mathrm{E} \neq \mathrm{B}$$
$$Bucket(D):\ \mathrm{D} \neq \mathrm{A}\ \ ||\ \ R_{DCB}$$
$$Bucket(C):\ \mathrm{C} \neq \mathrm{B}\ \ ||\ \ R_{ACB}$$
$$Bucket(B):\ \mathrm{B} \neq \mathrm{A}\ \ ||\ \ R_{AB}$$
$$Bucket(A):\ \ R_{A}$$



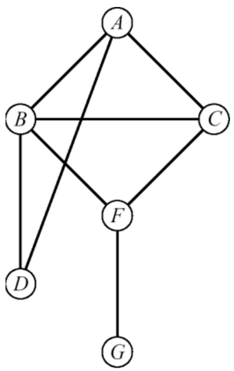

$$Bucket(A):\ \mathrm{A} \neq \mathrm{D},\ \mathrm{A} \neq \mathrm{B}$$
$$Bucket(D):\ \mathrm{D} \neq \mathrm{E}\ \ ||\ \ R_{DB}$$
$$Bucket(C):\ \mathrm{C} \neq \mathrm{B},\ \mathrm{C} \neq \mathrm{E}$$
$$Bucket(B):\ \mathrm{B} \neq \mathrm{E}\ \ ||\ \ R^{D}{}_{BE},\ R^{C}{}_{BE}$$
$$Bucket(E):\ \ ||\ R_{E}$$




**Complexity** **:** $O(n\ exp(w^{*}(d)))$,

$w^{*}(\mathrm{d})$ - ***induced width along ordering d***

# From bucket-elimination to bucket-tree propagation

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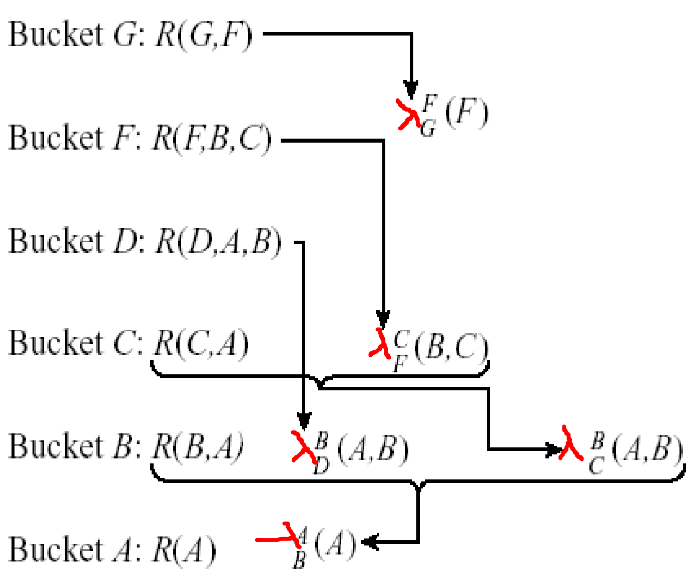


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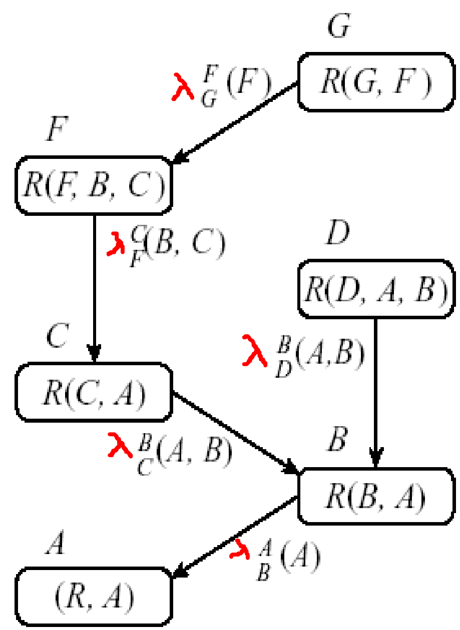
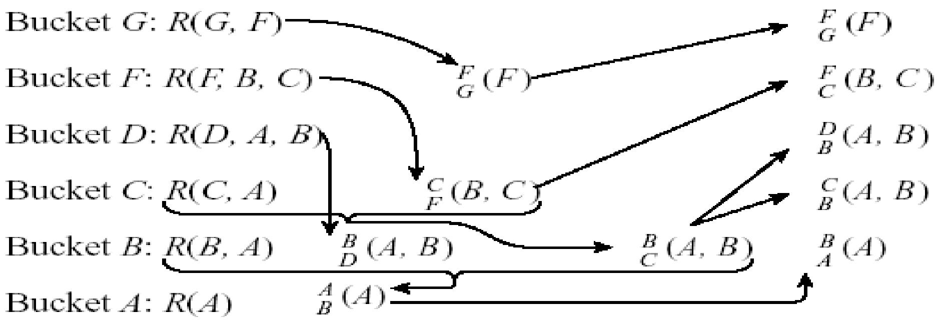
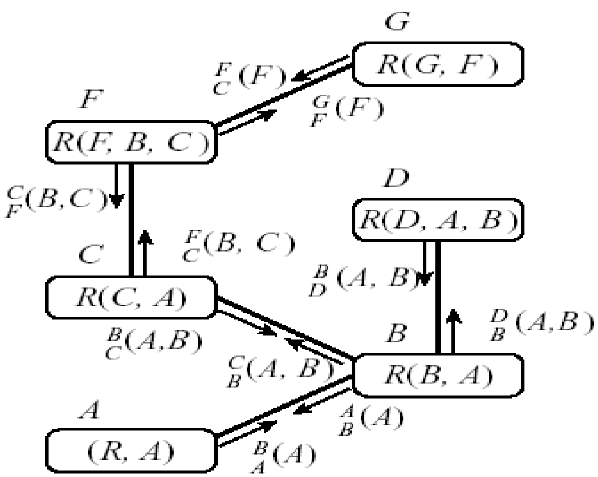
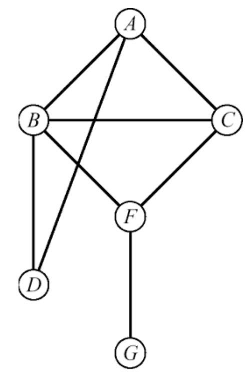



(b)

# The bottom up messages



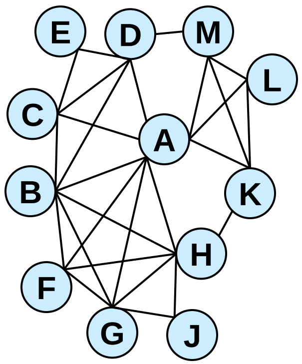

(a)

# Adaptive-Tree-Consistency as tree-decomposition

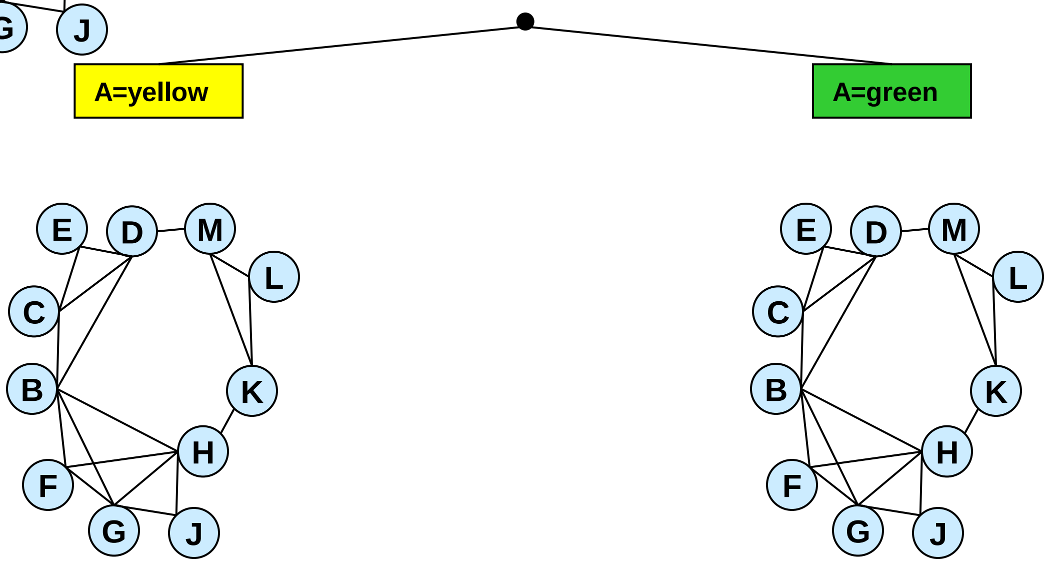
- Adaptive consistency is a message-passing along a bucket-tree
- **Bucket trees**: each bucket is a node and it is connected to a bucket to which its message is sent.
- **Theorem**: A bucket-tree is a tree-decomposition Therefore, CTE adds a bottom-up message passing to bucket-elimination.
- The complexity of ATC is O(r deg k^(w*+1)) time and O(n k^(sep*+1)) space.

## Conditioning

Graph
Coloring
problem

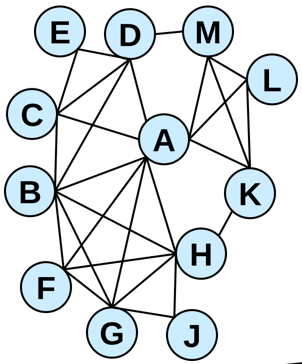



- Inference may require too much memory
- **Condition** on some of the variables



## Conditioning

Graph Coloring problem

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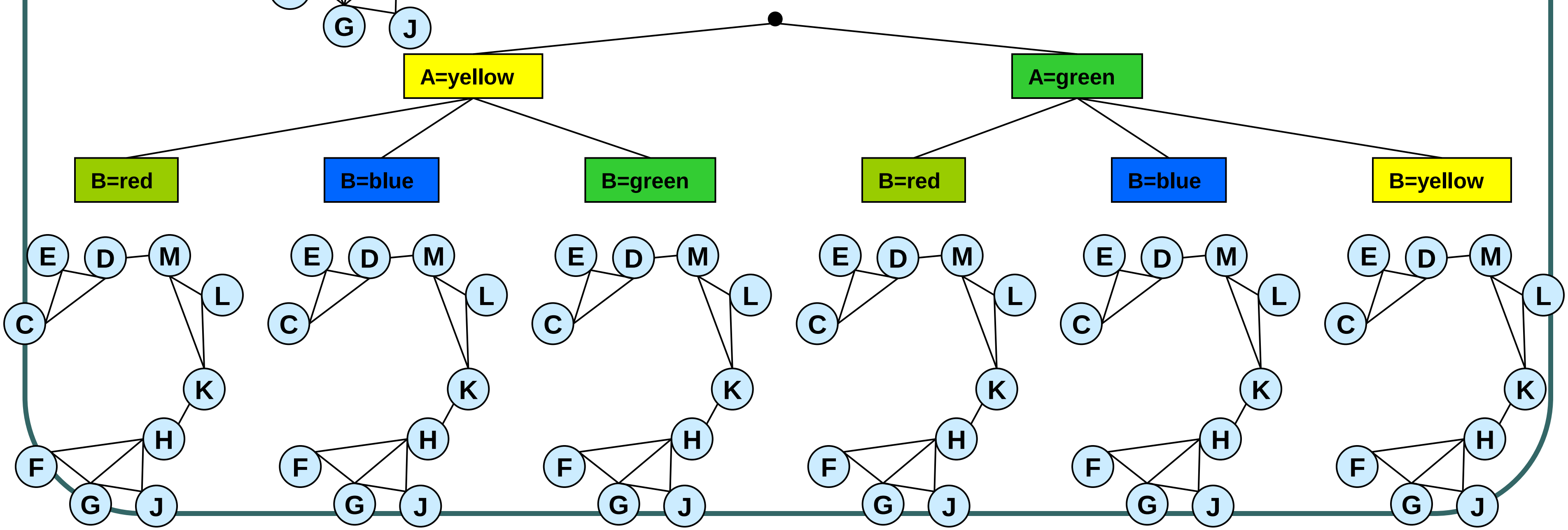
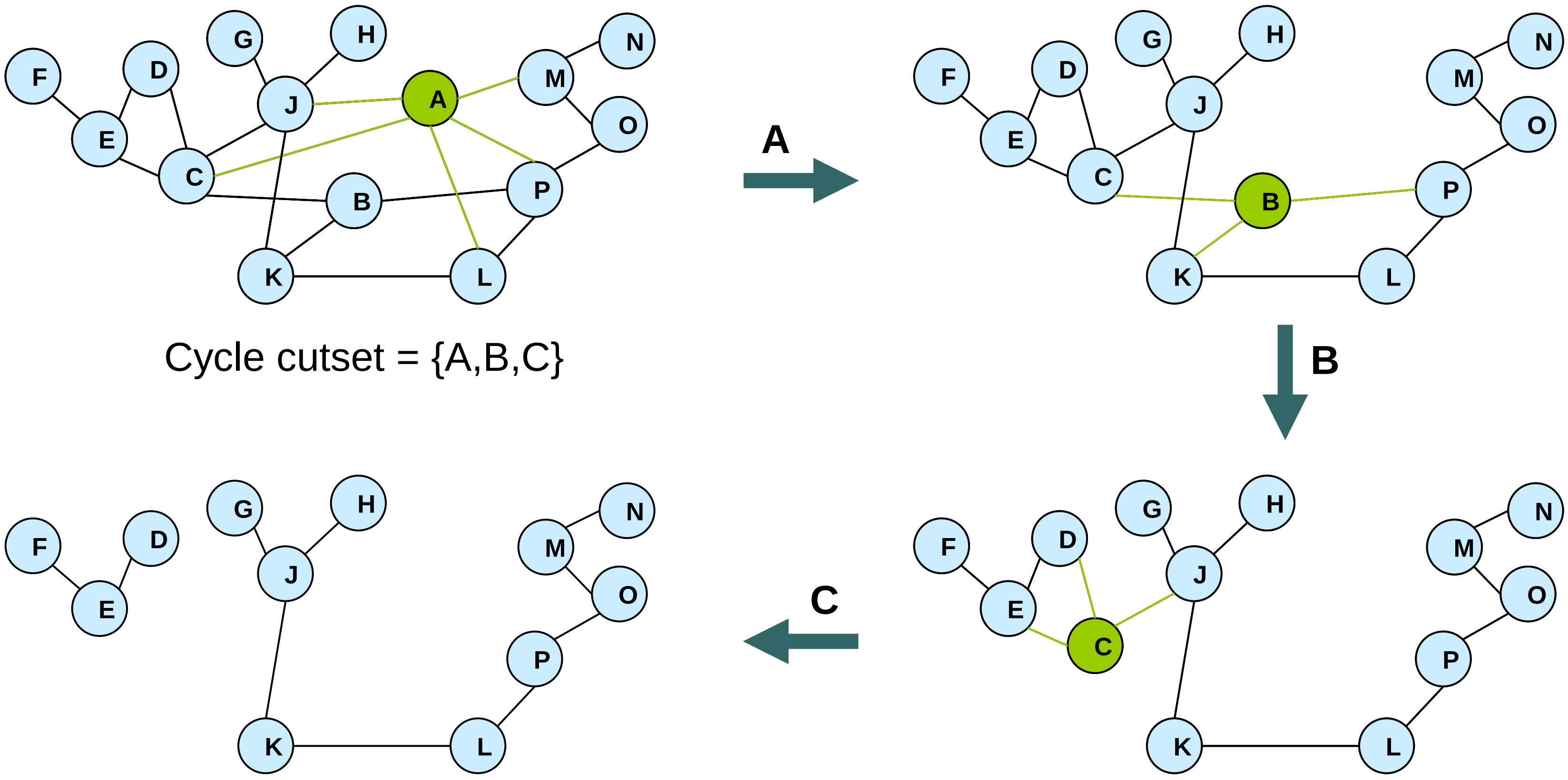


## Cycle cutset

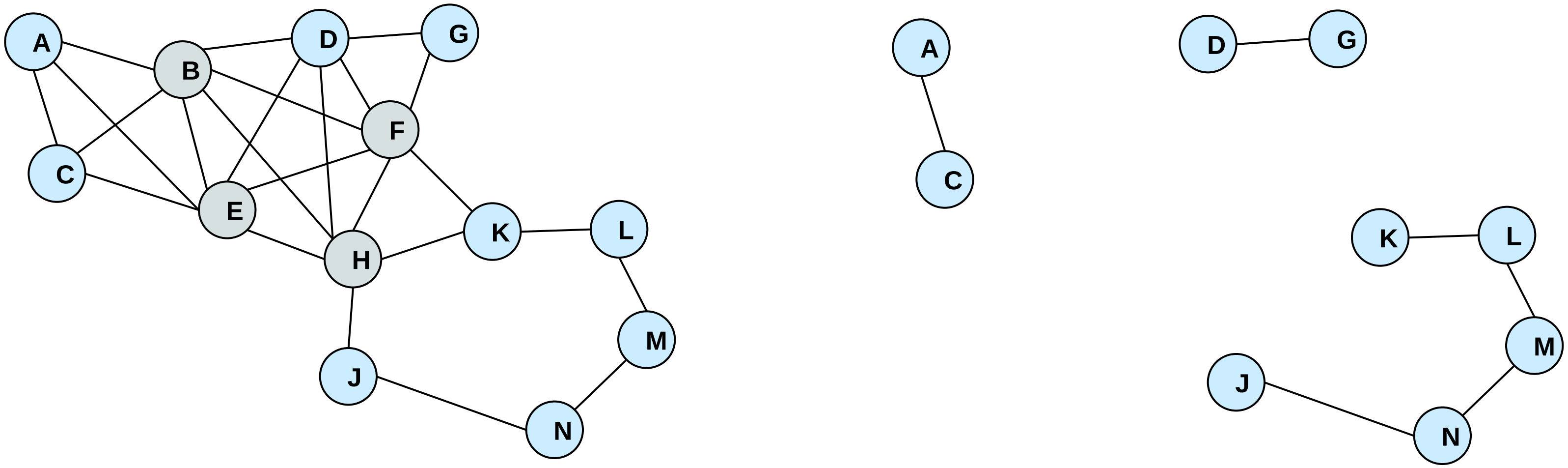



Cycle cutset = {A,B,C}

## Transforming into a tree

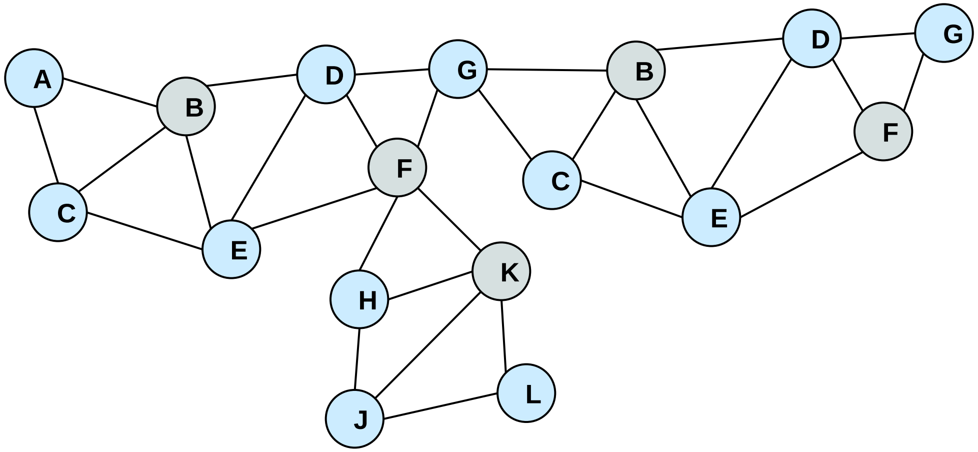
- **By Inference**
  - Time and spacde exponential in tree-width
- **By Conditioning-search**
  - Time exponential in the cycle-cutset

# Treewidth equals cycle cutset




treewidth = cycle cutset = 4

# Treewidth smaller than cycle cutset




treewidth = 2

cycle cutset = 5