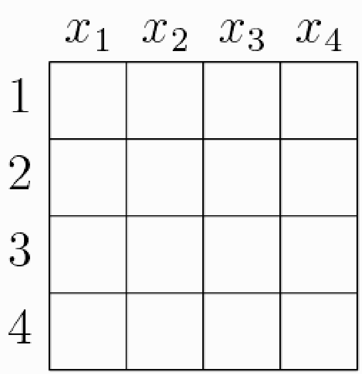
CONSTRAINT PROCESSING

Chapter 2

Figure 2.1: The 4-queens constraint network. The network has four variables, all with domains $D_i = \{1, 2, 3, 4\}$. (a) The labeled chess board. (b) The constraints between variables.

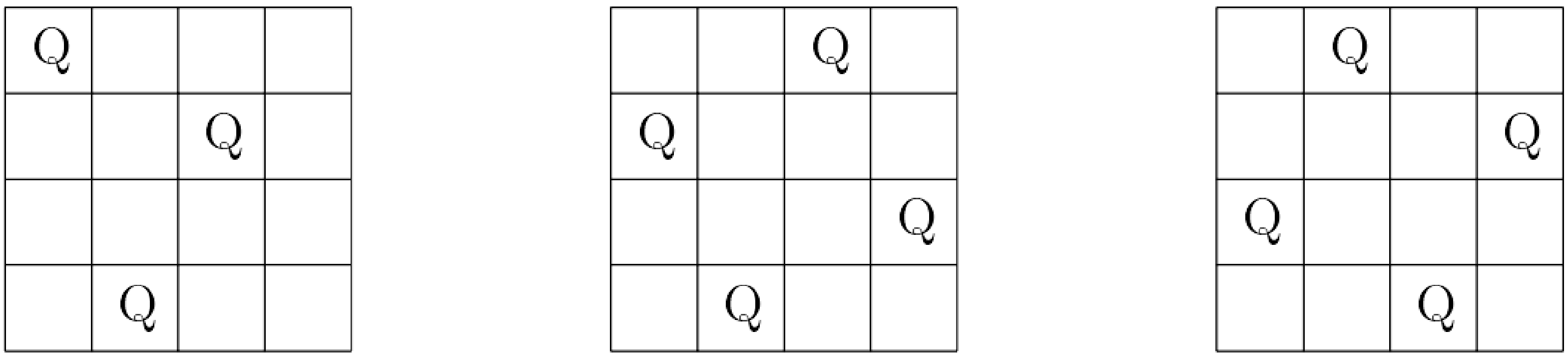


(a)

$$R_{12} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}$$
$$R_{13} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}$$
$$R_{14} = \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4) \\ (4,2), (4,3)\}$$
$$R_{23} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}$$
$$R_{24} = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}$$
$$R_{34} = \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}$$

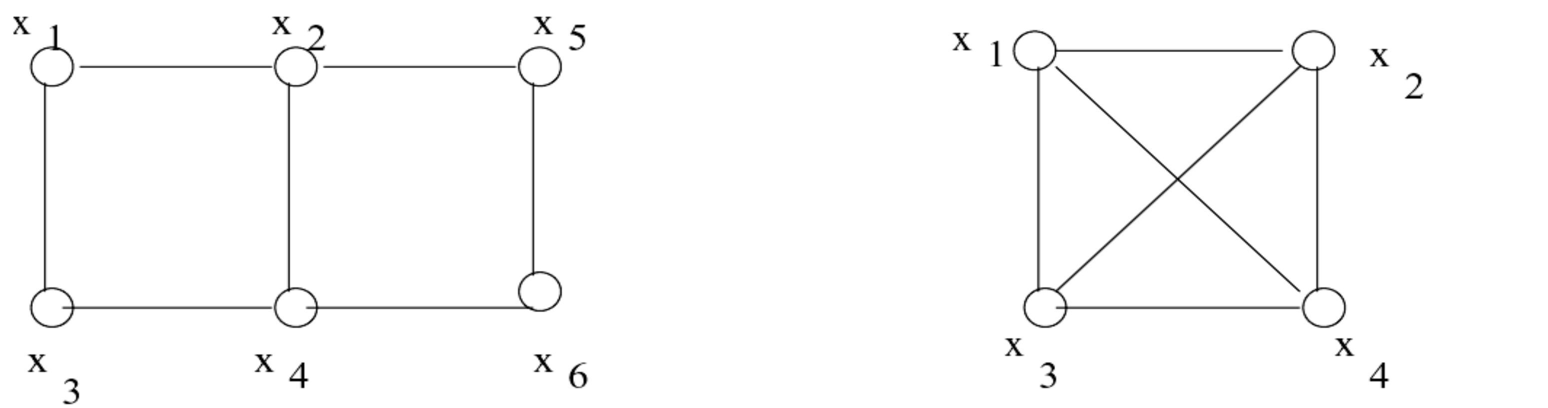
(b)

Figure 2.2: Not all consistent instantiations are part of a solution: (a) A consistent instantiation that is not part of a solution. (b) The placement of the queens corresponding to the solution (2, 4, 1, 3). (c) The placement of the queens corresponding to the solution (3, 1, 4, 2).



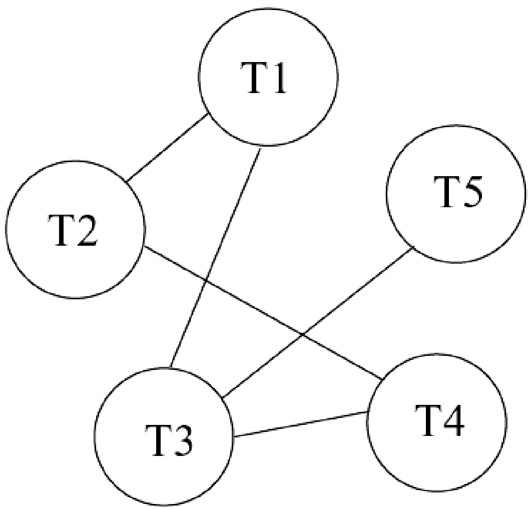
(a) (b) (c)

Figure 2.3: Constraint graphs of (a) the crossword puzzle and (b) the 4-queens problem.



(a) (b)

Figure 2.4: The constraint graph and constraint relations of the scheduling problem example.



Unary constraint

$D_{T4} = \{1{:}00,\ 3{:}00\}$

Binary constraints

$R_{\{T1,T2\}}$: {(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)}

$R_{\{T1,T3\}}$: {(2:00,1:00), (3:00,1:00), (3:00,2:00)}

$R_{\{T2,T4\}}$: {(1:00,2:00), (1:00,3:00), (2:00,1:00), (2:00,3:00), (3:00,1:00), (3:00,2:00)}

$R_{\{T3,T4\}}$: {(1:00,2:00), (1:00,3:00), (2:00,3:00)}

$R_{\{T3,T5\}}$: {(2:00,1:00), (3:00,1:00), (3:00,2:00)}

Figure 2.6: Constraint graphs of 3 instances of the Radio frequency assignment problem in CELAR's benchmark

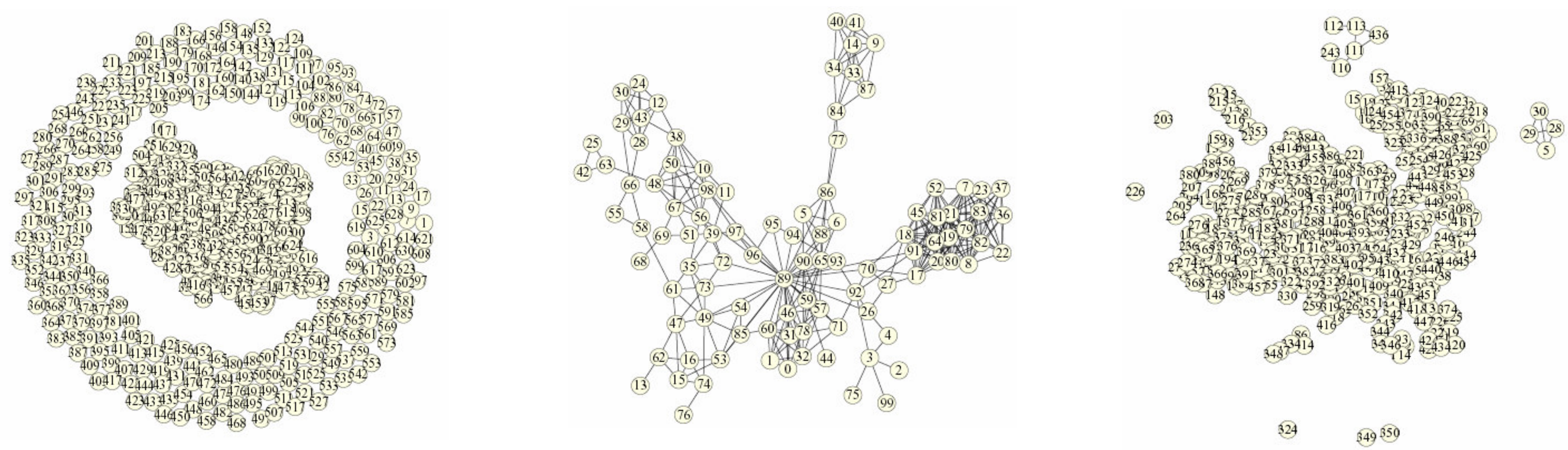
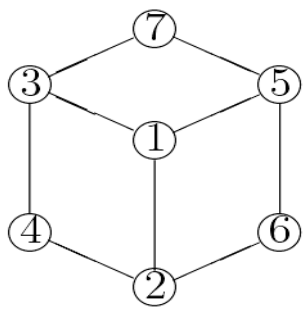


Figure 2.7: Scene labeling constraint network

$$R_{21} = \begin{bmatrix} 0&1&1&0&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \end{bmatrix} \quad R_{31} = \begin{bmatrix} 0&1&0&1&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \end{bmatrix} \quad R_{51} = \begin{bmatrix} 0&1&0&0&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \end{bmatrix}$$



$$R_{24} = R_{37} = R_{56} = \begin{bmatrix} 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&1&0 \end{bmatrix}$$

$$R_{26} = R_{34} = R_{57} = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&1 \end{bmatrix}$$

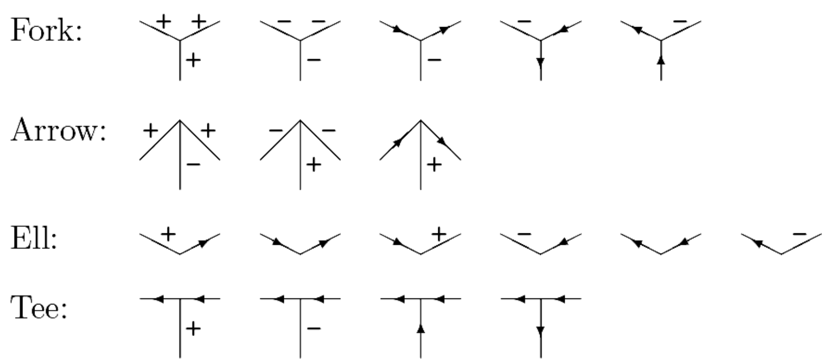


Figure 2.8: The interaction graph of theory $\varphi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$.

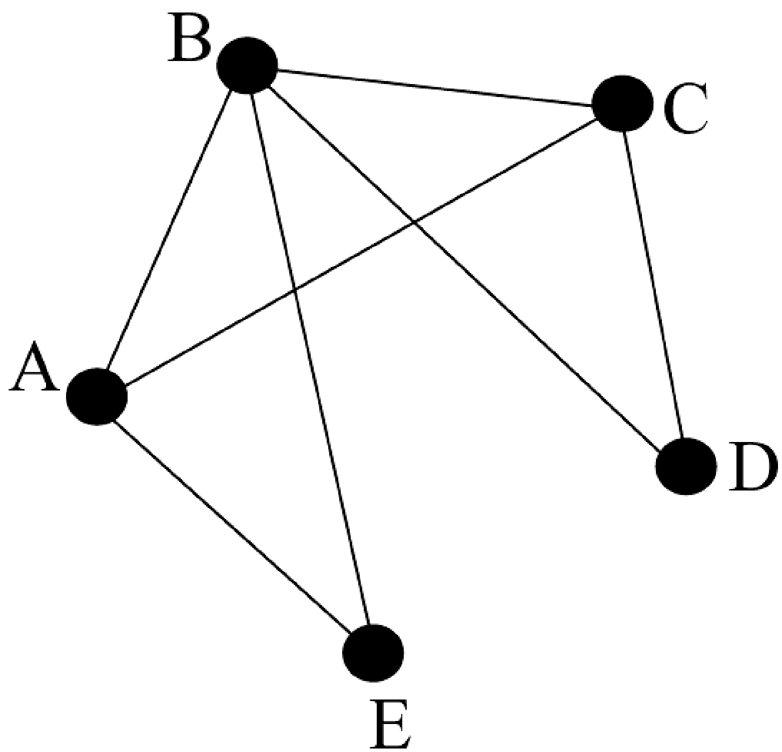
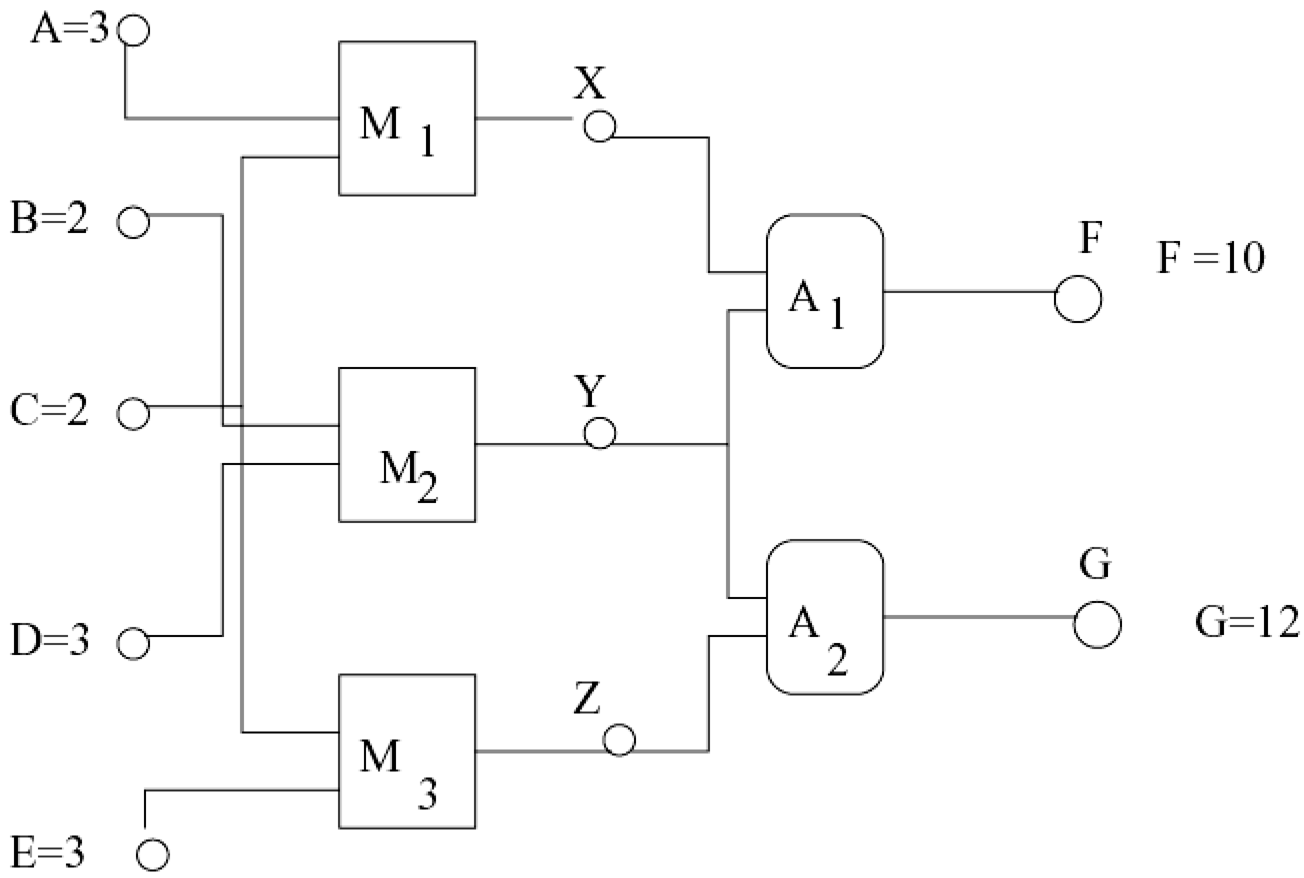
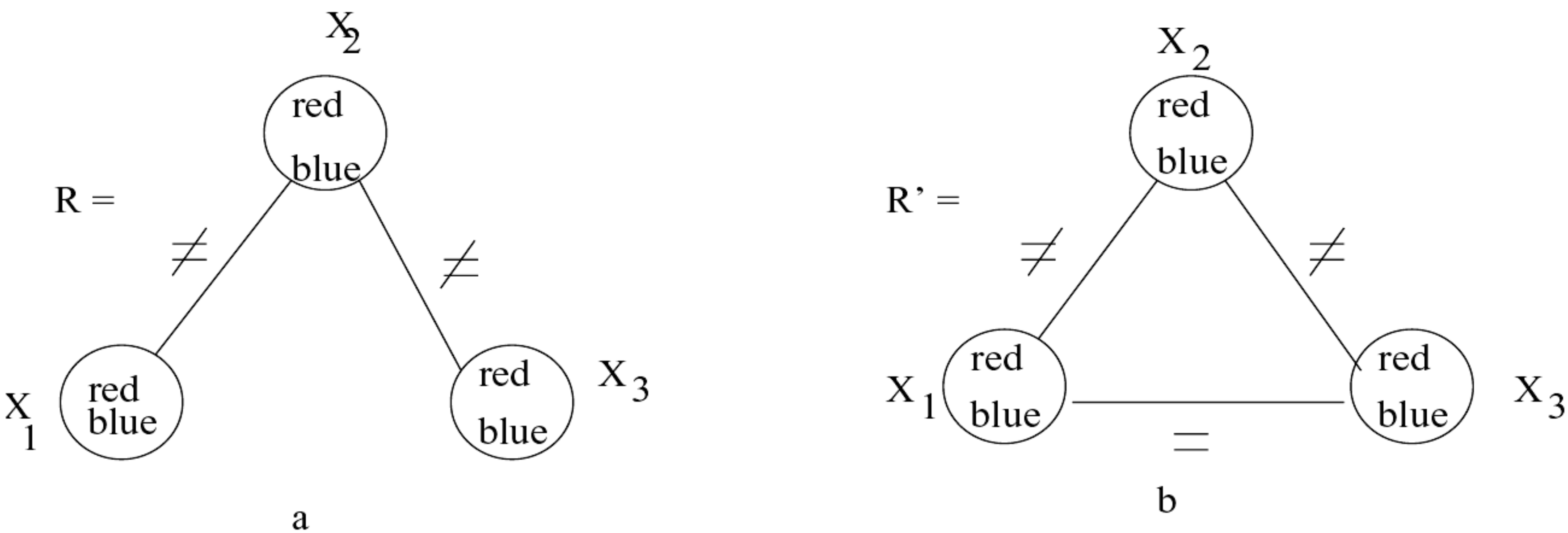


Figure 2.9: A combinatorial circuit: *M* is a multiplier, *A* is an adder.



Properties of binary constraint networks:

Figure 2.10: (a) A graph ℜ to be colored by two colors, (b) an equivalent representation ℜ' having a newly inferred constraint between x_1 and x_3.



Equivalence and deduction with constraints (composition)

Relations vs networks

- Can we represent the relations
- x1,x2,x3 = (0,0,0)(0,1,1)(1,0,1)(1,1,0)
- X1,x2,x3,x4 = (1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)

Relations vs networks

- Can we represent the relations
- x1,x2,x3 = (0,0,0)(0,1,1)(1,0,1)(1,1,0)
- X1,x2,x3,x4 = (1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)
- Most relations cannot be represented by networks:
- Number of relations 2^(n^k)
- Number of networks: 2^((k^2)(n^2))

The minimal and projection networks

- The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).
- *Relation = {(1,1,2)(1,2,2)(1,2,1)}*
  - *What is the projection network?*
- What is the relationship between a relation and its projection network?
- {(1,1,2)(1,2,2)(2,1,3)(2,2,2)}, solve its projection network?

Projection network (continued)

- **Theorem**: *Every relation is included in the set of solutions of its projection network.*

- **Theorem**: *The projection network is the tightest upper bound binary networks representation of the relation.*

Projection network

Theorem 2.3.8 *For every relation* ρ, $\rho \subseteq sol(P(\rho))$.

Theorem 2.3.9 *The projection network* $P(\rho)$ *is the tightest upper bound network representation of* ρ; *there is no binary network* $\mathcal{R}'$, *s.t.* $\rho \subseteq sol(R') \subset sol(P(\rho))$.

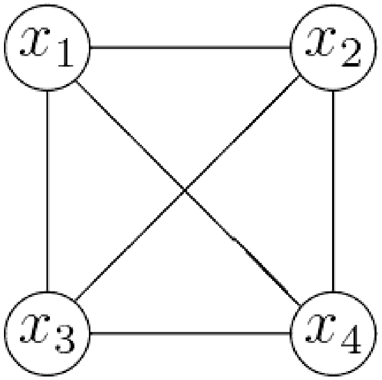
The Minimal Network
(partial order between networks)

Definition 2.3.10 *Given two binary networks, $\mathcal{R}'$ and $\mathcal{R}$, on the same set of variables $x_1, ..., x_n$, $\mathcal{R}'$ is* at least as tight as $\mathcal{R}$ *iff for every i and j, $R'_{ij} \subseteq R_{ij}$.*

Definition 2.3.14 *Let $\{\mathcal{R}_1, ...\mathcal{R}_l\}$ be the set of all networks equivalent to $\mathcal{R}_0$ and let $\rho = sol(\mathcal{R}_0)$. Then the minimal network M of $\mathcal{R}_0$ is defined by $M(\mathcal{R}_0) = \cap_{i=1}^{l} \mathcal{R}_i$.*

Theorem 2.3.15 *For every binary network $\mathcal{R}$ s.t. $\rho = sol(\mathcal{R})$, $M(\rho) = P(\rho)$.*

Figure 2.11: The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints.
(c) The minimal unary constraints (the domains).



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\} \\
M_{13} &= \{(2,1),\ (3,4)\} \\
M_{14} &= \{(2,3),\ (3,2)\} \\
M_{23} &= \{(1,4),\ (4,1)\} \\
M_{24} &= \{(1,2),\ (4,3)\} \\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
D_1 &= \{1,3\} \\
D_2 &= \{1,4\} \\
D_3 &= \{1,4\} \\
D_4 &= \{1,3\}
\end{aligned}
$$

(c)

Minimal network

- The minimal network is perfectly explicit for binary and unary constraints:
  - Every pair of values permitted by the minimal constraint is in a solution.
- Binary-decomposable networks:
  - A network whose all projections are binary decomposable
  - The minimal network repesenst fully binary-decomposable networks.
  - Ex: (x,y,x,t) = {(a,a,a,a)(a,b,b,b,)(b,b,a,c)} is binary representable but what about its projection on x,y,z?