On Planar Factor Graphs

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Motivated by the heuristics outlined in [YFW04] we show the following

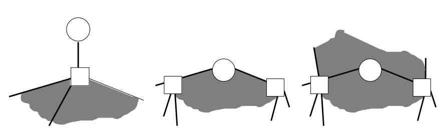
Proposition. *Every planar factor graph P for which no variable node has degree greater than 2 admits a region graph G with the following properties*

1. *The region graph includes regions corresponding to the shortest cycles in G*
2. *The sum of the counting numbers of all regions equals one*
3. *The region graph approximation is maxent-normal*

Proof: Fix an embedding of P in the plane. We construct the desirable region graph $G(P)$ from three types of regions. The largest, **face regions**, consist of the boundaries of the faces of the factor graph along with any additional variable nodes required to complete the region. We will include all face regions in $G(P)$ except for the "outside" face region corresponding to the unbounded face of the P. **Edge regions** contain a single factor node and its associated variable nodes and **vertex regions** consist of single variable nodes. To construct the region graph $G(P)$, add directed edges from each face region to all the edges it contains and similarly from edge regions to vertex regions. This region graph satisfies property #1 since the set of face regions includes the shortest cycles in P.

Before proceeding further, we establish that every vertex region has counting number 1 or 0. There are three cases:

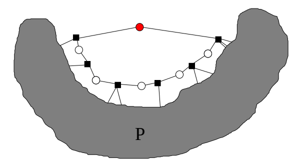
- The variable node is attached to a single factor node. In this case if the factor is contained in f faces, the vertex region corresponding to the variable node has f face ancestors and hence counting number $c_v = 1 - [f + (1 - f)] = 0$
- The variable node is adjacent to the "outside" face. Let the two attached verticies be contained in f_1 and f_2 faces respectively. Then the vertex region has $f_1 + f_2 - 1$ face regions ancestors (we subtract 1 to avoid double counting the adjacent face). This yields the counting number $c_v = 1 - [f_1 + f_2 - 1 + (1 - f_1) + (1 - f_2)] = 0$
- The variable node is adjacent to two faces. This gives the counting number $c_v = 1 - [f_1 + f_2 - 2 + (1 - f_1) + (1 - f_2)] = 1$. We have subtracted two to avoid overcounting the adjacent faces



To show #2 use induction on the number of face regions. Let $C(G(P))$ be the sum of the counting numbers of $G(P)$. First we show $C(G(P)) = 1$ for any P with zero face regions. $G(P)$ contains $|E|$ edge regions with counting number 1 and $|V|$ vertex regions each with counting number $1 - \deg(v)$ so

$$\begin{aligned} C(G(P)) &= |E| + \sum_{v \in V} (1 - \deg(v)) \\ &= |E| + |V| - \sum_{v \in V} \deg(v) \\ &= 1 \end{aligned}$$

since P is a tree with $|E| + |V|$ nodes and $\sum_{v \in V} \deg(v)$ edges.



Now suppose $C(G(P)) = 1$ for all P with some $n \geq 0$ face regions. For any P where $G(P)$ has $n+1$ face regions, choose a variable node in the "outside" face of P whose removal yields a connected subgraph P' with one fewer faces. $C(G(P')) = 1$ so we need to show that $C(G(P)) = C(G(P'))$. Since the variable is on the boundary, its removal will eliminate a single face region. We also know that the counting number of the removed vertex region is 0. Let k be the number of factor nodes lying on the boundary of the face. Each corresponding edge region looses a single face ancestor and hence each counting number goes up by 1. Since the graph is bipartite, there must also be $k-1$ variable nodes along the face boundary which had counting number 1 and now have counting number 0.[1] The net effect is no change in the total counting number as desired $C(G(P')) = C(G(P)) - 1 + k - (k-1) = C(G(P))$.

Lastly, we show #3, that G is maxent normal. Since all vertex regions have a counting number of 1 or 0, we may proceed as the argument for the Bethe approximation used in [YFW04].

$$\begin{aligned} H_G(X) &= \sum_{f \in F} H(X_f) + \sum_{e \in E} (1 - \deg(e)) H(X_e) + \sum_{v \in V} H(X_v) \\ &= \sum_{f \in F} \left(H(X_f) - \sum_{e : Pa(e) = f} H(X_e) \right) + \sum_{e \in E} H(X_e) + \sum_{v \in V} H(X_v) \\ &= -\sum_{f \in F} I(X_{e_1}; X_{e_2}; \ldots) + \sum_{e \in E} H(X_e) + \sum_{v \in V} H(X_v) \end{aligned}$$

where sums are taken only over those regions with non-zero counting number. All three terms on the right are maximized by the uniform distribution on X. ∎

[1] Although there may be other vertex regions which had the destroyed face region as an ancestor, their counting number is unaffected since their corresponding variable nodes border on the same number of faces before and after the deletion.

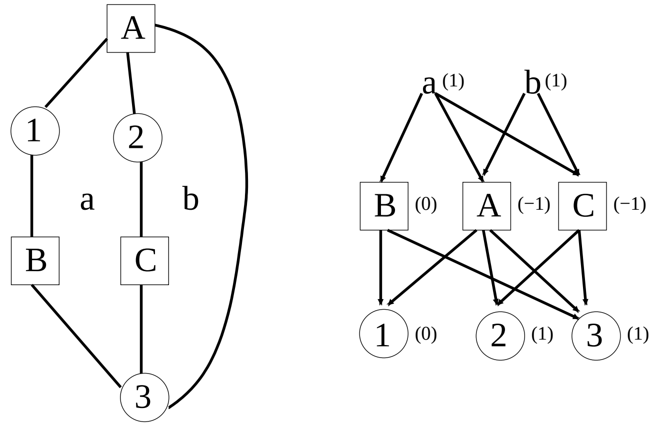


Figure 1: A counter example for high degree variable nodes, $\sum c_R = 2$.

The proof of #2 is just Euler's formula (F-E+V=2) extend to hypergraphs where planarity is taken to be planarity of the equivalent bipartite graph and we have removed the "outside" face. One might be tempted (as I was) to think that this construction would work for all planar factor graphs. Unfortunately it is not to be. The figure above provides a counter-example with a degree three variable node. In such a case, some other region graph construction would be prefered.

[YFW04] J. S. Yedidia, W. T. Freeman, and Yair Weiss "Constructing Free Energy Approximations and Generalized Belief Propagation Algorithms," MERL Tech Report TR-2004-040, May 2004.